Precalculus

Third Edition (3.0)

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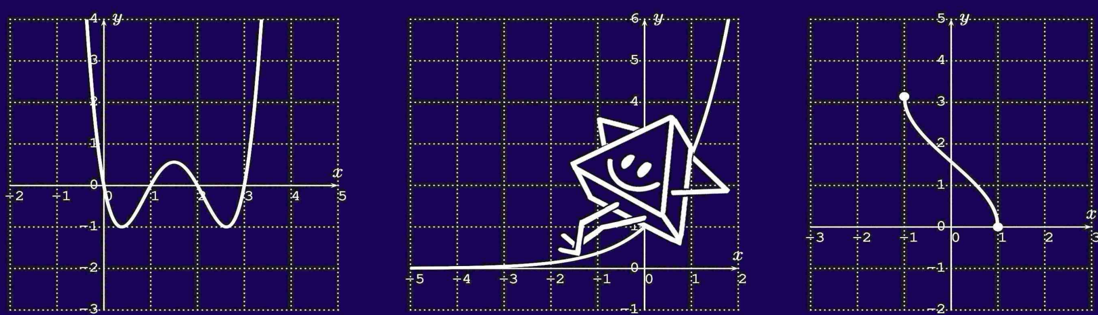
Precalculus Textbook Download Page



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Chapter 1

Intervals and functions

In this chapter, we will give a definition of the main topic of this course, the notion of a function. Before we introduce functions, we review some notation regarding sets of numbers in Section 1.1.

1.1 Review of number sets

We start with a brief review of number systems and intervals.

Review 1.1: Number systems

The **natural numbers** (denoted by $\mathbb{N}$) are the numbers

$$1, 2, 3, 4, 5, \ldots$$

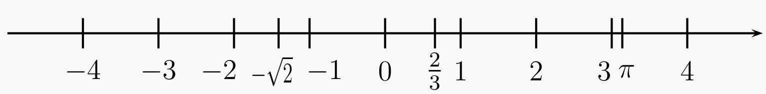
The **integers** (denoted by $\mathbb{Z}$) are the numbers

$$\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \ldots$$

The **rational numbers** (denoted by $\mathbb{Q}$) are the fractions $\frac{a}{b}$ of integers a and b with $b \neq 0$. Here are some examples of rational numbers:

$$\frac{3}{5}, -\frac{2}{6}, 17, 0, \frac{3}{-8}$$

The **real numbers** (denoted by $\mathbb{R}$) are the numbers on the real number line



Here are some examples of real numbers:

$$\sqrt{3}, \pi, -\frac{2}{5}, 18, 0, 6.789$$

A real number that is not a rational number is called an **irrational number**. Here are some examples of irrational numbers:

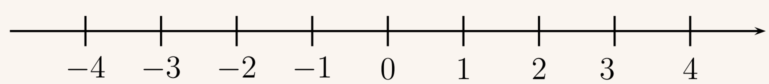
$$\pi, \sqrt{2}, 5^{\frac{2}{3}}, e$$

Recall that there is an order relation on the set of real numbers:

$4 < 9$	reads as	4 is less than 9,
$-3 \leq 2$	reads as	-3 is less than or equal to 2,
$\frac{7}{6} > 1$	reads as	$\frac{7}{6}$ is greater than 1,
$2 \geq -3$	reads as	2 is greater than or equal to -3.

Note 1.2

- Note that $2 < 3$, but $-2 > -3$, which can easily be seen on the number line.

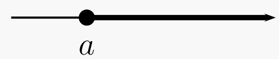
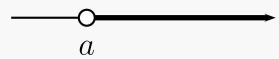


- Note that $5 \leq 5$ and $5 \geq 5$. However the same is not true when using the symbol $<$. We write this as $5 \not< 5$.

We also review some basic notation concerning intervals.

Review 1.3: Intervals

The set of all real numbers x greater than or equal to some number a and/or less than or equal to some number b is a subset of the real numbers, which is an interval. There are several ways to write an interval: in interval notation, graph it on the number line, or write it as an inequality.

Inequality notation	Number line	Interval notation
$a \leq x \leq b$		$[a, b]$
$a < x < b$		(a, b)
$a \leq x < b$		$[a, b)$
$a < x \leq b$		$(a, b]$
$a \leq x$		$[a, \infty)$
$a < x$		(a, ∞)
$x \leq b$		$(-\infty, b]$
$x < b$		$(-\infty, b)$

Formally, we define the interval $[a, b]$ to be the set of all real numbers x such that $a \leq x \leq b$:

$$[a, b] \quad = \quad \{ \ x \ | \ a \leq x \leq b \ \}$$

There are similar definitions for the other intervals shown in the above table.

Be careful!

Be sure to write the smaller number $a < b$ first when writing an interval $[a, b]$. For example, the interval

$$[5, 3] = \{ \ x \ | \ 5 \leq x \leq 3 \ \} = \{\}$$

would be the empty set!

Example 1.4

Write the inequality

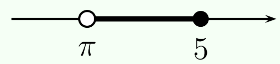
$$\pi < x \leq 5$$

in interval notation and graph it on the number line.

Solution.

Interval notation: $(\pi, 5]$

On the number line:



$\square$

Example 1.5

Write the interval

as an inequality and in interval notation.

Solution.

Inequality notation: $-3 \leq x$

Interval notation: $[-3, \infty)$

$\square$

Example 1.6

Write the interval

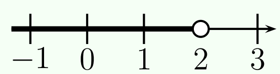
$$(-\infty, 2)$$

as an inequality and graph it on the number line.

Solution.

Inequality notation: $x < 2$

On the number line:



$\square$

Note 1.7: Number line notation

In some texts round and square brackets are also used on the number line to depict an interval. For example, the following depicts the interval $[2, 5)$.

1.2 Introduction to functions

We now introduce the notion of a function. An easy and well-known example of a function is given by the equation of a straight line such as, for example, $y = 5x + 4$. Note that for each given x we obtain an induced y. (For example, for $x = 3$, we obtain $y = 5 \cdot 3 + 4 = 19$.)

Definition 1.8: Function, domain, codomain

A **function** f consists: a set D of **inputs** called the **domain**, a set C of possible **outputs** called the **codomain**, and an assignment that assigns to each input x exactly one output y.
A function f with domain D and codomain C is denoted by

$$f : D \to C.$$

If x is in the domain D (an input), then we denote by $f(x) = y$ the output that is assigned by f to x.

Sometimes it is of interest to know the set of all elements in the codomain that actually occur as an output. This set is a subset of the codomain and is called the range. We have:

Definition 1.9: Range

The **range** R of a function f is a subset of the codomain of f, given by all of the outputs of f:

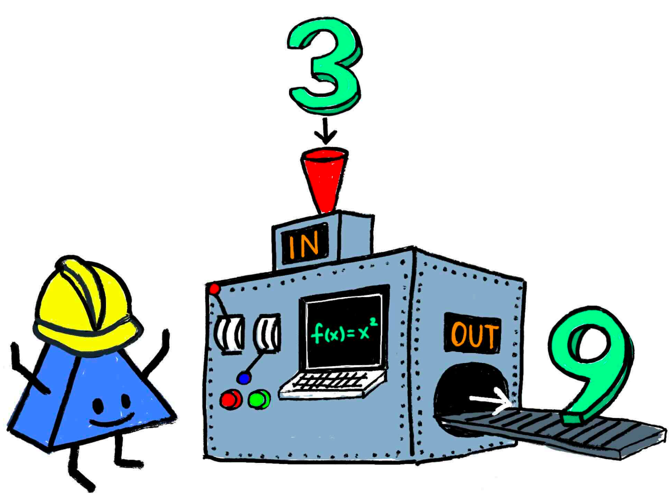
$$R = \{f(x) \,|\, x \text{ is in the domain of } f\}$$

Note 1.10

Some authors use a slightly different convention in that they use the term range for what we called the codomain above.

Since we will be dealing with many functions it is convenient to name various functions (usually with letters f, g, h, etc.). Often we will implicitly assume that a domain and codomain are given without specifying these explicitly. If the range can be determined and the codomain is not given explicitly, then we take the codomain to be the range. If the range cannot easily be determined and the codomain is not explicitly given, then the codomain should be taken to be a set which clearly contains the range. For example, in many instances the codomain can be taken to be the set of all real numbers.

There are many ways that one can describe how a function assigns to an input an output (all of which may not apply to a specific function): via a table of values (listing the input-output pairs); via a formula (with the domain and range explicitly or implicitly given); via a graph (representing input-output pairs on a coordinate plane); or in words, just to name a few. Examples of these ways to represent a function will be given below.



Example 1.11

Define the assignment f by the following table

x	2	5	-3	0	7	4
y	6	8	6	4	-1	8

The assignment f assigns to the input 2 the output 6, which is also written as

$$f(2) = 6.$$

Similarly, f assigns to 5 the number 8, in short $f(5) = 8$, etc.:

$$f(5) = 8, \quad f(-3) = 6, \quad f(0) = 4, \quad f(7) = -1, \quad f(4) = 8.$$

The domain D is the set of all inputs. The domain is therefore

$$D = \{-3, 0, 2, 4, 5, 7\}.$$

The range R is the set of all outputs. The range is therefore

$$R = \{-1, 4, 6, 8\}.$$

The assignment f is indeed a function since each element of the domain gets assigned exactly one element in the range. Note that for an input number that is not in the domain, f does not assign an output to it. For example,

$$f(1) = \text{undefined}.$$

Note also that $f(5) = 8$ and $f(4) = 8$, so that f assigns to the inputs 5 and 4 the same output 8. Similarly, f also assigns the same output to the inputs 2 and -3. Therefore we see that:

- *A function may assign the same output to two different inputs!*

Example 1.12

Consider the assignment f that is given by the following table.

x	2	5	-3	0	5	4
y	6	8	6	4	-1	8

This assignment *does not define* a function! What went wrong? Consider the input value 5. What does f assign to the input 5? The third column states that f assigns to 5 the output 8, whereas the sixth column states that f assigns to 5 the output -1,

$$f(5) = 8, \qquad f(5) = -1.$$

However, by the definition of a function, to each input we have to assign *exactly one* output. So here, to the input 5 we have assigned two outputs 8 and -1. Therefore, f is not a function.

- *A function cannot assign two outputs to one input!*

We repeat the two bullet points from the last two examples, which are crucial for the understanding of a function.

Note 1.13: Same output from inputs versus multiple outputs

- *A function may assign the same output to two different inputs!*

  $f(x_1) = y$ and $f(x_2) = y$ with $x_1 \neq x_2$ is **allowed**!

- *A function cannot assign two outputs to one input!*

  $f(x) = y_1$ and $f(x) = y_2$ with $y_1 \neq y_2$ is **not allowed**!

Example 1.14

A university creates a mentoring program which matches each freshman student with a senior student as his or her mentor. Within this program it is guaranteed that each freshman gets precisely one mentor, however two freshmen may receive the same mentor. Does the assignment of freshmen to mentor, or mentor to freshmen describe a function? If so, what is its domain, what is its range?

Solution.

Since a senior may mentor several freshman, we cannot take a mentor as an “input,” as he or she would be assigned to several “output” freshmen students. So freshman is not a function of mentor.

On the other hand, we can assign each freshmen to exactly one mentor, which therefore describes a function.
The domain (the set of all inputs) is given by the set of all freshmen students. The range (the set of all outputs) is given by the set of all senior students that are mentors. The function assigns each "input" freshmen student to his or her unique "output" mentor. □

Example 1.15

The rainfall in a city for each of the 12 months is displayed in the following histogram.

a) Is the rainfall a function of the month?

b) Is the month a function of the rainfall?

Solution.

a) Each month has exactly one amount of rainfall associated to it. Therefore, the assignment that associates to a month its rainfall (in inches) is a function.

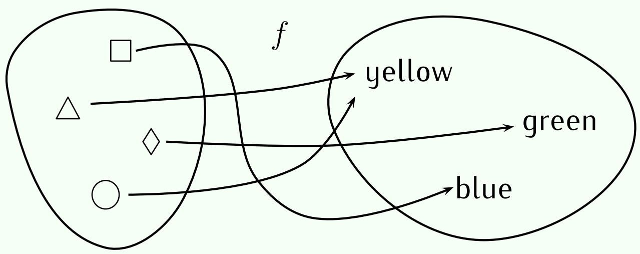
b) If we take a certain rainfall amount as our input data, can we associate a unique month to it? For example, February and March appear to have the same amount of rainfall. If the months February and March do indeed have the same amount of rainfall, then, to one input amount of rainfall we cannot assign a unique month. The month would therefore *not* be a function of the rainfall.

   One could also argue that the rainfall for any two months in the above graph would almost certainly be different if the rainfall is measured to a high enough degree of accuracy, as it is very unlikely that any two months have the exact amount of rainfall. (What does it even mean to have the exact same amount of rainfall?) For this setup, one would conclude that the month *is* a function of the rainfall.

□

Example 1.16

Consider the function f described below.



Here, the function f maps the input symbol $\square$ to the output color blue. Other assignments of f are as follows:

$$f(\square) = \text{blue} \qquad f(\triangle) = \text{yellow}$$
$$f(\Diamond) = \text{green} \qquad f(\bigcirc) = \text{yellow}$$

The domain is the set of symbols $D = \{\square, \triangle, \Diamond, \bigcirc\}$, and the range is the set of colors $R = \{\text{blue, green, yellow}\}$. Notice, in particular, that the inputs $\triangle$ and $\bigcirc$ both have the same output yellow, which is certainly allowed for a function.

Example 1.17

Consider the function $y = 5x+4$ with domain all real numbers and range all real numbers. Note that for each input x, we obtain an exactly one induced output y. For example, for the input $x = 3$ we get the output $y = 5 \cdot 3 + 4 = 19$, etc.

Example 1.18

Consider the function $y = x^2$ with domain all real numbers and range non-negative numbers. The function takes a real number as an input and squares it. For example if $x = -2$ is the input, then $y = 4$ is the output.

Example 1.19

For each real number x, denote by $\lfloor x \rfloor$ the greatest integer that is less or equal to x. We call $\lfloor x \rfloor$ the **floor of** x. For example, to calculate $\lfloor 4.37 \rfloor$, note that all integers $4, 3, 2, \dots$ are less than or equal to 4.37:

$$\dots, -3, -2, -1, 0, 1, 2, 3, 4 \qquad \leq \qquad 4.37$$

The greatest of these integers is 4, so that $\lfloor 4.37 \rfloor = 4$. We define the **floor function** as $f(x) = \lfloor x \rfloor$. Here are more examples of function values of the floor function.

$$\lfloor 7.3 \rfloor = 7 \qquad \lfloor \pi \rfloor = 3 \qquad \lfloor -4.65 \rfloor = -5$$
$$\lfloor 12 \rfloor = 12 \qquad \left\lfloor \frac{-26}{3} \right\rfloor = \lfloor -8.667 \rfloor = -9$$

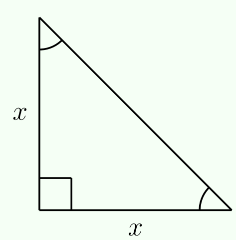
The domain of the floor function is the set of all real numbers, that is $D = \mathbb{R}$. The range is the set of all integers, $R = \mathbb{Z}$.

Example 1.20

Let A be the area of an isosceles right triangle with base side length x. Express A as a function of x.

Solution.

Being an isosceles right triangle means that two side lengths are x, and the angles are $45°$, $45°$, and $90°$ (or in radian measure $\frac{\pi}{4}$, $\frac{\pi}{4}$, and $\frac{\pi}{2}$):



Recall that the area of a triangle is: area $= \frac{1}{2}$ base $\cdot$ height. In this case, we have base$= x$, and height$= x$, so that the area

$$A = \frac{1}{2} x \cdot x = \frac{1}{2} x^2.$$

Therefore, the area $A(x) = \frac{1}{2} \cdot x^2$. □

Example 1.21

Consider the equation $y = x^2 + 3$. This equation associates to each input number a exactly one output number $b = a^2 + 3$. Therefore, the equation defines a function. For example:

To the input 5 we assign the output $5^2 + 3 = 25 + 3 = 28$.

The domain D is all real numbers, $D = \mathbb{R}$. Since x^2 is always ≥ 0, we see that $x^2 + 3 \geq 3$, and vice versa, every number $y \geq 3$ can be written as $y = x^2 + 3$. (To see this, note that the input $x = \sqrt{y-3}$ for $y \geq 3$ gives the output $x^2 - 3 = (\sqrt{y-3})^2 + 3 = y - 3 + 3 = y$.) Therefore, the range is $R = [3, \infty)$.

Example 1.22

Consider the equation $x^2 + y^2 = 25$. Does this equation define y as a function of x? That is, does this equation assign to each input x exactly one output y?
An input number x gets assigned to y with $x^2 + y^2 = 25$. Solving this for y, we obtain

$$y^2 = 25 - x^2 \qquad \Longrightarrow \qquad y = \pm\sqrt{25 - x^2}.$$

Therefore, there are *two* possible outputs associated to the input $x(\neq 5)$:

$$\text{either} \quad y = +\sqrt{25 - x^2} \qquad \text{or} \quad y = -\sqrt{25 - x^2}.$$

For example, the input $x = 0$ has two outputs $y = 5$ and $y = -5$. However, a function cannot assign two outputs to one input x! The conclusion is that $x^2 + y^2 = 25$ does *not* determine y as a function!

Note 1.23: Independent versus dependent variable

Note that if $y = f(x)$ then x is called the **independent variable** and y is called the **dependent variable** (since x can be chosen freely, and y depends on x).
If $x = g(y)$ then y is the independent variable and x is the dependent variable (since now, y can be chosen freely and x depends on y).

1.3 Exercises

Exercise 1.1

Give examples of numbers that are

a) natural numbers
b) integers
c) integers but not natural numbers
d) rational numbers
e) real numbers
f) rational numbers but not integers

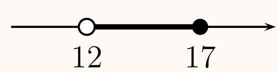
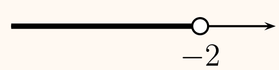
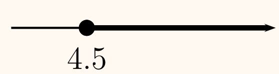
Exercise 1.2

Which of the following numbers are natural numbers, integers, rational numbers, or real numbers? Which of these numbers are irrational?

a) $\frac{7}{3}$ b) -5 c) 0 d) $17,000$ e) $\frac{12}{4}$ f) $\sqrt{7}$ g) $\sqrt{25}$

Exercise 1.3

Complete the table.

Inequality notation	Number line	Interval notation
$2 \leq x < 5$		
$x \leq 3$		
		
		
		$[-2, 6]$
		$(-\infty, 0)$
		
$5 < x \leq \sqrt{30}$		
		$(\frac{13}{7}, \pi)$

Exercise 1.4

The tables below describe assignments between inputs x and outputs y. Determine which of the given tables describe a function. If they do, determine their domain and range. Describe which outputs are assigned to which inputs.

a)

x	-5	3	-1	6	0
y	5	2	8	3	7

b)

x	6	17	4	-2	4
y	8	-2	0	3	-1

c)

x	19	7	6	-2	3	-11
y	3	3	3	3	3	3

d)

x	1	2	3	3	4	5
y	5.33	9	13	13	17	$\sqrt{19}$

e)

x	0	1	2	2	3	4
y	0	1	2	3	3	4

Exercise 1.5

In a store, every item that is for sale has a price.

a) Does the assignment which assigns to an item its price constitute a function (in the sense of Definition 1.8 on page 6)?

b) Does the assignment which assigns to a given price all items with this price constitute a function?

c) In the case where the assignment is a function, what is the domain?

d) In the case where the assignment is a function, what is the range?

Exercise 1.6

A bank offers wealthy customers a certain amount of interest if they keep more than 1 million dollars in their account. The amount is described in the following table.

dollar amount x in the account	interest amount
$x \leq \$1,000,000$	$\$0$
$\$1,000,000 < x \leq \$10,000,000$	2% of x
$\$10,000,000 < x$	1% of x

a) Justify that the assignment cash amount to interest defines a function.

b) Find the interest for an amount of:

i) $\$50,000$ ii) $\$5,000,000$ iii) $\$1,000,000$
iv) $\$30,000,000$ v) $\$10,000,000$ vi) $\$2,000,000$

Exercise 1.7

Find a formula for a function describing the given inputs and outputs.

a) *input*: the radius of a circle
output: the circumference of the circle

b) *input*: the side length in an equilateral triangle
output: the perimeter of the triangle

c) *input*: one side length of a rectangle, with other side length being 3
output: the perimeter of the rectangle

d) *input*: the side length of a cube
output: the volume of the cube