Geometric series

Lesson #25

MAT 1375 Precalculus

New York City College of Technology CUNY

Geometric sequences

Geometric sequence

An **geometric sequence** is a sequence where each term follows from its previous term by multiplying a fixed number r:

$$a_1 \xrightarrow{\cdot r} a_2 \xrightarrow{\cdot r} a_3 \xrightarrow{\cdot r} a_4 \xrightarrow{\cdot r} a_5 \qquad \dots$$

r is called the **common ratio**. A geometric sequence has the closed form formula:

$$\boxed{a_n = a_1 \cdot r^{n-1}}$$

Example

1. Find the first 5 terms of the sequence $a_n = 3 \cdot 10^{n-1}$.
   Answer:
   $$3, 30, 300, 3000, 30000, \dots$$
   This is a geometric sequence with first term $a_1 = 3$ and common ratio $r = 10$.
2. Find the rule of the geometric sequence: $5, 10, 20, 40, 80, 160, \dots$
   Answer:
   $$a_n = 5 \cdot 2^{n-1}$$

Geometric sequence

$$a_n = a_1 \cdot r^{n-1}$$

Find the closed form formula for the given geometric sequence.

1. $7, 14, 28, 56, \ldots$
   $a_1 = 7, r = 2, \Rightarrow a_n = 7 \cdot 2^{n-1}$
2. $4, -20, 100, -500, \ldots$
   $a_1 = 4, r = -5, \Rightarrow a_n = 4 \cdot (-5)^{n-1}$
3. $8, 0.8, 0.08, 0.008, \ldots$
   $a_1 = 8, r = \frac{1}{10}, \Rightarrow a_n = 8 \cdot \left(\frac{1}{10}\right)^{n-1}$
4. $48, 24, 12, 6, 3, \frac{3}{2}, \frac{3}{4}, \ldots$
   $a_1 = 48, r = \frac{1}{2}, \Rightarrow a_n = 48 \cdot \left(\frac{1}{2}\right)^{n-1}$

Find the stated term of the given geometric sequence.

1. 14th term of $5, -10, 20, -40, 80, \ldots$
   $a_n = 5 \cdot (-2)^{n-1}$
   $\Rightarrow a_{14} = 5 \cdot (-2)^{13} = 5 \cdot (-8192) = -40,960$
2. 17th term of $3, 6, 12, 24, 48, \ldots$
   $a_n = 3 \cdot 2^{n-1}$
   $\Rightarrow a_{17} = 3 \cdot 2^{16} = 196,608$
3. 12th term of $54, 18, 6, 2, \ldots$
   $a_n = 54 \cdot \left(\frac{1}{3}\right)^{n-1}$
   $\Rightarrow a_{12} = 54 \cdot \left(\frac{1}{3}\right)^{11} = \frac{2 \cdot 27}{3^{11}} = \frac{2 \cdot 3^3}{3^{11}} = \frac{2}{3^8} = \frac{2}{6561}$
4. 10th term of $24, 36, 54, 81, \ldots$
   $r = \frac{36}{24} = \frac{3}{2}, \Rightarrow a_n = 24 \cdot \left(\frac{3}{2}\right)^{n-1}$
   $\Rightarrow a_{10} = 24 \cdot \left(\frac{3}{2}\right)^9 = \frac{24 \cdot 3^9}{2^9} = \frac{472,392}{512} = \frac{59,049}{64}$
5. 7th term of $75, -30, 12, -4.8, \ldots$
   $r = \frac{-30}{75} = -\frac{2}{5}, \Rightarrow a_n = 75 \cdot \left(-\frac{2}{5}\right)^{n-1}$
   $\Rightarrow a_7 = 75 \cdot \left(-\frac{2}{5}\right)^6 = \frac{75 \cdot 2^6}{5^6} = \frac{4,800}{15,625} = \frac{192}{625}$

Example

Find the sum of the first 4 terms of $3, 30, 300, 3000, 30000, 300000, \ldots$.

Answer: Multiply by $(1-10)$ and use the "telescopic sum" trick:

$$\begin{aligned}(1-10)\cdot(3+30+300+3000) \\ &= 3 \underbrace{-\cancel{30}+\cancel{30}} \underbrace{-\cancel{300}+\cancel{300}} \underbrace{-\cancel{3000}+\cancel{3000}} -30000 \\ &= 3-30000\end{aligned}$$

Therefore: $(1-10)\cdot\sum_{n=1}^{4} a_n = 3-30000 \qquad \Rightarrow \sum_{n=1}^{4} a_n = \frac{3-30000}{1-10} = 3\cdot\frac{1-10000}{1-10}$.

Evaluating this gives the expected answer: $\sum_{n=1}^{4} a_n = 3\cdot\frac{-9999}{-9} = 3\cdot 1111 = 3333$.

Geometric series

For any geometric series $\quad a_n = a_1\cdot r^{n-1} \quad$ the sum of the first p terms is

$$\boxed{\sum_{n=1}^{p} a_n = a_1\cdot\frac{1-r^p}{1-r}}$$

Geometric series

For $a_n = a_1 \cdot r^{n-1}$:

$$\sum_{n=1}^{p} a_n = a_1 \cdot \frac{1-r^p}{1-r}$$

1. Find $\sum_{n=1}^{9} 4 \cdot 6^{n-1}$

   $p = 9, \quad a_1 = 4, \quad r = 6$

   $\Rightarrow \sum_{n=1}^{9} 4 \cdot 6^{n-1} = 4 \cdot \frac{1-6^9}{1-6}$

   $= \frac{4 \cdot (-10{,}077{,}695)}{-5} = 8,062,156$

2. Find $\sum_{n=1}^{6} 5 \cdot (-3)^{n-1}$

   $p = 6, \quad a_1 = 5, \quad r = -3$

   $\Rightarrow \sum_{n=1}^{6} 5 \cdot (-3)^{n-1}$

   $= 5 \cdot \frac{1-(-3)^6}{1-(-3)} = 5 \cdot \frac{1-729}{1+3}$

   $= 5 \cdot \frac{-728}{4} = -910$

Find the sum of the given geometric sequence.

1. Sum the first 7 terms of $2, -10, 50, -250, \ldots$

   We need $\sum_{n=1}^{7} a_n$ for $a_n = 2 \cdot (-5)^{n-1}$.

   $p = 7, \quad a_1 = 2, \quad r = -5$

   $\Rightarrow \sum_{n=1}^{7} a_n = 2 \cdot \frac{1-(-5)^7}{1-(-5)} = 2 \cdot \frac{1+78125}{1+5} = 26,042$

2. Sum the first 8 terms of $9, 6, 4, \frac{8}{3}, \ldots$

   We need $\sum_{n=1}^{8} a_n$ for $a_n = 9 \cdot (\frac{2}{3})^{n-1}$.

   $p = 8, \quad a_1 = 9, \quad r = \frac{2}{3}$

   $\Rightarrow \sum_{n=1}^{8} a_n = 9 \cdot \frac{1-(\frac{2}{3})^8}{1-\frac{2}{3}} = \frac{9 \cdot (1-\frac{256}{6561})}{\frac{1}{3}}$

   $= 9 \cdot \frac{6561-256}{6561} \cdot \frac{3}{1} = \frac{9 \cdot 6205 \cdot 3}{6561} = \frac{6205}{243}$

3. Sum the first 15 terms of $7, -\frac{7}{3}, \frac{7}{9}, -\frac{7}{27}, \ldots$

   We need $\sum_{n=1}^{15} a_n$ for $a_n = 7 \cdot (-\frac{1}{3})^{n-1}$.

   $p = 15, \quad a_1 = 7, \quad r = -\frac{1}{3}$

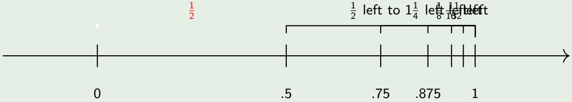
   $\Rightarrow \sum_{n=1}^{15} a_n = 7 \cdot \frac{1-(-\frac{1}{3})^{15}}{1-(-\frac{1}{3})} = \frac{7(1+\frac{1}{3^{15}})}{\frac{4}{3}} = 7 \cdot \frac{3^{15}+1}{3^{15}} \cdot \frac{3}{4}$

   $= \frac{7 \cdot 14{,}348{,}908}{3^{14} \cdot 4} = \frac{7 \cdot 3{,}587{,}227}{3^{14}} = \frac{25{,}110{,}589}{4{,}782{,}969}$

Example

Add all infinitely many terms in the geometric series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \dots$.

Answer:



Therefore: $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots = 1$

Infinite geometric series

For any geometric sequence $a_1, \quad a_1 \cdot r, \quad a_1 \cdot r^2, \quad a_1 \cdot r^3, \quad \dots$
where the common ratio r is between $-1 < r < 1$, then the sum of all terms is

$$\boxed{\sum_{n=1}^{\infty} a_n = a_1 \cdot \frac{1}{1-r}}$$

Formal reason for this: $\sum_{n=1}^{\infty} a_n = \lim_{p\to\infty} \sum_{n=1}^{p} a_n = \lim_{p\to\infty} a_1 \cdot \frac{1-r^p}{1-r} = a_1 \cdot \frac{1}{1-r}$.

Note: If the common ratio $r \geq 1$ or $r \leq -1$, then the infinite sum **does not exist**!
Example: The sum $5 + 10 + 20 + 40 + 80 + \dots$ does not exist!

Infinite geometric series

For $a_1, \quad a_1 r, \quad a_1 r^2, \quad a_1 r^3, \ldots:$

$$\boxed{\sum_{n=1}^{\infty} a_n = a_1 \cdot \frac{1}{1-r}}$$

1. Find $\sum_{n=1}^{\infty} 6 \cdot \left(\frac{1}{3}\right)^{n-1}$

   $a_1 = 6, \ r = \frac{1}{3}$

   $\Rightarrow \sum_{n=1}^{\infty} a_n = 6 \cdot \frac{1}{1-\frac{1}{3}} = \frac{6}{\frac{2}{3}} = 6 \cdot \frac{3}{2} = \frac{18}{2} = 9$

2. Find $\sum_{n=1}^{\infty} 20 \cdot \left(\frac{2}{5}\right)^{n-1}$

   $a_1 = 20, \ r = \frac{2}{5}$

   $\Rightarrow \sum_{n=1}^{\infty} a_n = 20 \cdot \frac{1}{1-\frac{2}{5}} = \frac{20}{\frac{3}{5}} = 20 \cdot \frac{5}{3} = \frac{100}{3}$

3. Find $\sum_{n=1}^{\infty} 24 \cdot \left(-\frac{1}{2}\right)^{n-1}$

   $a_1 = 24, \ r = -\frac{1}{2}$

   $\Rightarrow \sum_{n=1}^{\infty} a_n = 24 \cdot \frac{1}{1-(-\frac{1}{2})} = \frac{24}{\frac{3}{2}} = 24 \cdot \frac{2}{3} = \frac{48}{3} = 16$

Find the sum of the given geometric sequence.

1. Sum all terms of $12, 3, \frac{3}{4}, \frac{3}{16}, \ldots$

   $a_1 = 12, \ r = \frac{1}{4}$

   $\Rightarrow \sum_{n=1}^{\infty} a_n = 12 \cdot \frac{1}{1-\frac{1}{4}} = \frac{12}{\frac{3}{4}} = 12 \cdot \frac{4}{3} = 16$

2. Sum all terms of $-27, 9, -3, 1, \ldots$

   $a_1 = -27, \ r = -\frac{1}{3}$

   $\Rightarrow \sum_{n=1}^{\infty} a_n = -27 \cdot \frac{1}{1-(-\frac{1}{3})} = \frac{-27}{\frac{4}{3}} = -27 \cdot \frac{3}{4} = -\frac{81}{4}$

3. Sum all terms of $50, -30, 18, \ldots$

   $a_1 = 50, \ r = \frac{-30}{50} = -\frac{3}{5}$

   $\Rightarrow \sum_{n=1}^{\infty} a_n = 50 \cdot \frac{1}{1-(-\frac{3}{5})} = \frac{50}{\frac{8}{5}} = 50 \cdot \frac{5}{8} = \frac{250}{8} = \frac{125}{4}$

4. Sum all terms of $36, 24, 16, \ldots$

   $a_1 = 36, \ r = \frac{24}{36} = \frac{2}{3}$

   $\Rightarrow \sum_{n=1}^{\infty} a_n = 36 \cdot \frac{1}{1-\frac{2}{3}} = \frac{36}{\frac{1}{3}} = 36 \cdot \frac{3}{1} = 108$

5. Sum all terms of $0.4, 0.04, 0.004, 0.0004, \ldots$

   $a_1 = 0.4, \ r = 0.1$

   $\Rightarrow \sum_{n=1}^{\infty} a_n = 0.4 \cdot \frac{1}{1-0.1} = \frac{0.4}{0.9} = \frac{4}{9} = 0.444\overline{4}$