Arithmetic sequences and series

Lesson #24

MAT 1375 Precalculus

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Sequences

Definition

A **sequence** is an ordered list of numbers.

$$a_1, a_2, a_3, a_4, a_5, \ldots$$

Here a_1 is the first number, a_2 is the second number, a_3 is the third number, etc.

Examples of sequences:

- $5, 8, 11, 14, 17, 20, 23, \ldots$ (arithmetic sequence) $a_1 = 5,\ a_2 = 8,\ a_3 = 11$
- $3, 6, 12, 24, 48, 96, 192, \ldots$ (geometric sequence)
- $1, 4, 9, 16, 25, 36, 49, \ldots$ (sequence of squares)
- $7, -3.6, \sqrt{2}, \frac{2}{3}, \pi, -22, \ldots$ (no apparent rule)
- $1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$ (Fibonacci sequence)

Find the first 5 terms of the sequence:

1. $a_n = 4n + 5$
   $9, 13, 17, 21, 25, \ldots$
2. $a_n = \sqrt{2n+3}$
   $\sqrt{5}, \sqrt{7}, 3, \sqrt{11}, \sqrt{13}, \ldots$
3. $a_n = n^2$
   $1, 4, 9, 16, 25, \ldots$
4. $a_n = \frac{n+2}{n+3}$
   $\frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{7}{8}, \ldots$

Find the first 6 terms of the sequence given by the recursive rule.

1. $a_1 = 5, \quad a_n = 2 \cdot a_{n-1}$
   $5, 10, 20, 40, 80, 160 \ldots$
   (geometric sequence)
2. $a_1 = 2, \quad a_n = a_{n-1} + 6$
   $2, 8, 14, 20, 26, 32 \ldots$
   (arithmetic sequence)
3. $a_1 = 1, a_2 = 1$
   $a_n = a_{n-1} + a_{n-2}$
   $1, 1, 2, 3, 5, 8 \ldots$
   (Fibonacci sequence)

Arithmetic sequence

An **arithmetic sequence** is a sequence where each term follows from its previous term by adding a fixed number d:

$$a_1 \xrightarrow{+d} a_2 \xrightarrow{+d} a_3 \xrightarrow{+d} a_4 \xrightarrow{+d} a_5 \quad \dots$$

d is called the **common difference**. An arithmetic sequence has the closed form formula:

$$\boxed{a_n = a_1 + (n-1) \cdot d}$$

Example

1. Find the first 5 terms of the sequence $a_n = 7 + (n-1) \cdot 3$.
   Answer:
   $$7, 10, 13, 16, 19, \dots$$
   This is an arithmetic sequence with first term $a_1 = 7$ and common difference $d = 3$.
2. Find the rule of the arithmetic sequence: $6, 11, 16, 21, 26, 31, \dots$
   Answer:
   $$a_n = 6 + (n-1) \cdot 5$$

Arithmetic sequence

$$a_n = a_1 + (n-1) \cdot d$$

Find the closed form formula for the given arithmetic sequence.

1. $8, 14, 20, 26, \ldots$
   $a_n = 8 + (n-1) \cdot 6$
2. $15, 22, 29, 36, \ldots$
   $a_n = 15 + (n-1) \cdot 7$
3. $13, 10, 7, 4, 1, -2, -5, \ldots$
   $a_n = 13 + (n-1) \cdot (-3)$
4. $-19, -11, -3, 5, 13, 21, \ldots$
   $a_n = -19 + (n-1) \cdot 8$

Find the stated term of the given arithmetic sequence.

1. 500th term of $8, 12, 16, 20, 24, \ldots$
   $a_n = 8 + (n-1) \cdot 4$
   $\Rightarrow a_{500} = 8 + 499 \cdot 4 = 2004$
2. 444th term of $4, 7, 10, 13, 16, \ldots$
   $a_n = 4 + (n-1) \cdot 3$
   $\Rightarrow a_{444} = 4 + 443 \cdot 3 = 1333$
3. 57th term of $5, -1, -7, -13, -20, \ldots$
   $a_n = 5 + (n-1) \cdot (-6)$
   $\Rightarrow a_{57} = 5 + 56 \cdot (-6) = -331$
4. 1234th term of $-5, -7, -9, -11, -13, \ldots$
   $a_n = -5 + (n-1) \cdot (-2)$
   $\Rightarrow a_{1234} = -5 + 1233 \cdot (-2) = -2471$

Series

Definition

A **series** is a sum of terms in a sequence. We use the following $\sum$ "sigma" notation:

$$\sum_{n=1}^{p} a_n = a_1 + \cdots + a_p$$

More generally: $\sum_{n=m}^{p} a_n = a_m + \cdots + a_p.$

1. $\sum_{n=1}^{4}(2n+3) =$
   $= 5 + 7 + 9 + 11$
   $= 32$

2. $\sum_{n=1}^{6}(9-n) =$
   $= 8 + 7 + 6 + 5 + 4 + 3$
   $= 33$

3. $\sum_{n=1}^{5}(n^2+2n) =$
   $= (1+2) + (4+4) + (9+6)$
   $+(16+8) + (25+10)$
   $= 3 + 8 + 15 + 24 + 35 = 85$

4. $\sum_{n=1}^{4} \frac{2}{n+3} =$
   $= \frac{2}{4} + \frac{2}{5} + \frac{2}{6} + \frac{2}{7} = \frac{1}{2} + \frac{2}{5} + \frac{1}{3} + \frac{2}{7}$
   $= \frac{105+84+70+60}{2\cdot 5\cdot 3\cdot 7} = \frac{319}{210}$

Example

Find the sum of the first 100 terms of $2, 4, 6, 8, 10, 12, \ldots$.

Answer: Use the "Gauss trick" (Carl Friedrich Gauss, 1777-1855):

$$\begin{array}{ccccccc} 2 & +4 & +6 & \ldots & +196 & +198 & +200 \\ 200 & +198 & +196 & \ldots & +6 & +4 & +2 \\ \hline 202 & +202 & +202 & \ldots & +202 & +202 & +202 \end{array}$$

Therefore: $2 \cdot \sum_{n=1}^{100} a_n = 100 \cdot 202 \quad \Rightarrow \sum_{n=1}^{100} a_n = \frac{100}{2} \cdot 202 = 50 \cdot 202 = 10100.$

Arithmetic series

For any arithmetic sequence $a_n = a_1 + (n-1) \cdot d$ the sum of the first p terms is

$$\boxed{\sum_{n=1}^{p} a_n = \frac{p}{2} \cdot (a_1 + a_p)}$$

Arithmetic series

For $a_n = a_1 + (n-1)\cdot d$:

$$\sum_{n=1}^{p} a_n = \frac{p}{2}\cdot(a_1 + a_p)$$

1. Find $\sum_{n=1}^{300}(7+(n-1)\cdot 5)$
   $p = 300, \quad a_1 = 7,$
   $a_{300} = 7 + 299\cdot 5 = 1502$
   $\Rightarrow \sum_{n=1}^{300}(7+(n-1)\cdot 5)$
   $= \frac{300}{2}\cdot(7+1502) = 226,350$
2. Find $\sum_{n=1}^{555}(-3n+9)$
   $p = 555, \quad a_1 = 6,$
   $a_{555} = -3\cdot 555 + 9 = -1656$
   $\Rightarrow \sum_{n=1}^{555}(-3n+9)$
   $= \frac{555}{2}\cdot(6+(-1656)) = -457,875$

Find the sum of the given arithmetic sequence.

1. Sum the first 97 terms of $11, 15, 19, 23, \ldots$
   We need $\sum_{n=1}^{97} a_n$ for $a_n = 11 + (n-1)\cdot 4$.
   $p = 97,\ a_1 = 11,\ a_{97} = 11 + 96\cdot 4 = 395$
   $\Rightarrow \sum_{n=1}^{97} a_n = \frac{97}{2}\cdot(11+395) = 19,691$
2. Sum the first 234 terms of $-17, -5, 7, 19, \ldots$
   We need $\sum_{n=1}^{234} a_n$ for $a_n = -17 + (n-1)\cdot 12$.
   $p = 234,\ a_1 = -17,\ a_{234} = -17 + 233\cdot 12 = 2,779$
   $\Rightarrow \sum_{n=1}^{234} a_n = \frac{234}{2}\cdot(-17+2779) = 323,154$
3. Sum the first 500 terms of $18, 13, 8, 3, -2, \ldots$
   We need $\sum_{n=1}^{500} a_n$ for $a_n = 18 + (n-1)\cdot(-5)$.
   $p = 500,\ a_1 = 18,\ a_{500} = 18 + 499\cdot(-5) = -2,477$
   $\Rightarrow \sum_{n=1}^{500} a_n = \frac{500}{2}\cdot(18+(-2,477)) = -614,750$