Complex numbers

Lesson #23

MAT 1375 Precalculus

New York City College of Technology CUNY

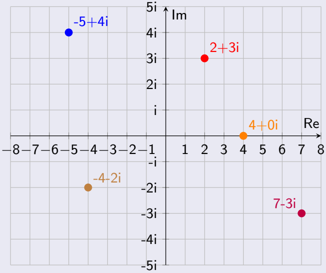
Complex numbers

A **complex number** is a number

$$a + b \cdot i, \quad \text{where } i^2 = -1$$

where a =real part, $b \cdot i$ =imaginary part.

Complex plane:



Plot in the complex plane:
$2+3i$, $-5+4i$, $7-3i$, $-4-2i$, $4 = 4+0i$

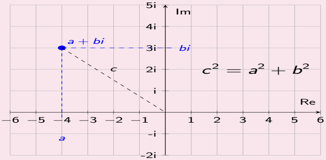
Perform the indicated operation:

1. $(2+3i)+(-6+5i) = -4+8i$
2. $(2+3i)-(-6+5i)$
   $= 2+3i+6-5i$
   $= 8-2i$
3. $(2+3i)\cdot(-6+5i)$
   $= -12+10i-18i+15i^2$
   $= -12-8i-15$
   $= -27-8i$
4. $\frac{2+3i}{-6+5i}$
   $= \frac{(2+3i)(-6-5i)}{(-6+5i)(-6-5i)}$
   $= \frac{-12-10i-18i-15i^2}{36+30i-30i-25i^2}$
   $= \frac{-12-28i+15}{36+25}$
   $= \frac{3-28i}{61}$
   $= \frac{3}{61} - \frac{28}{61} i$

The modulus (or absolute value) of a complex number

Modulus of a complex numbers

The **modulus** or **absolute value** of $a + bi$ is



$$|a + bi| = \sqrt{a^2 + b^2}$$

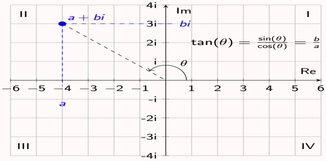
Find the modulus of the complex number.

1. $3 + 7i$
   $|3 + 7i| = \sqrt{3^2 + 7^2} = \sqrt{9 + 49} = \sqrt{58}$
2. $4 + 2i$
   $|4 + 2i| = \sqrt{4^2 + 2^2} = \sqrt{16 + 4}$
   $= \sqrt{20} = \sqrt{4 \cdot 5} = 2\sqrt{5}$
3. $-3 + 4i$
   $|-3 + 4i| = \sqrt{(-3)^2 + 4^2}$
   $= \sqrt{9 + 16} = \sqrt{25} = 5$
4. $5 - 5i$
   $|5 - 5i| = \sqrt{5^2 + (-5)^2} = \sqrt{25 + 25}$
   $= \sqrt{50} = 5\sqrt{2}$
5. $-4 - 4i$
   $|-4 - 4i| = \sqrt{(-4)^2 + (-4)^2}$
   $= \sqrt{16 + 16} = \sqrt{32} = \sqrt{16 \cdot 2} = 4\sqrt{2}$
6. $2 - 2\sqrt{3}i$
   $|2 - 2\sqrt{3}i| = \sqrt{2^2 + (-2\sqrt{3})^2}$
   $= \sqrt{4 + 4 \cdot 3} = \sqrt{4 + 12} = \sqrt{16} = 4$
7. $5\sqrt{3} + 5i$
   $|5\sqrt{3} + 5i| = \sqrt{(5\sqrt{3})^2 + 5^2}$
   $= \sqrt{25 \cdot 3 + 25} = \sqrt{100} = 10$

The argument (or angle) of a complex number

Argument of a complex numbers

The **argument** or **angle** of $a + bi$ is



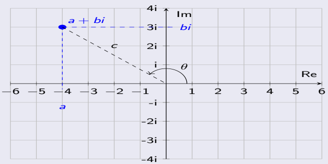
$\theta = \arctan(\frac{b}{a})$	if in I or IV
$\theta = \arctan(\frac{b}{a}) + 180^\circ$	if in II or III

Find the argument of the complex number.

1. $5 + 2i$
   $\theta = \arctan(\frac{2}{5}) \approx 21.8^\circ$
2. $-5 - 2i$ in quadrant III
   $\theta = \arctan(\frac{-2}{-5}) + 180^\circ$
   $\approx 21.8^\circ + 180^\circ \approx 201.8^\circ$
3. $3 + 5i$ in quadrant I
   $\theta = \arctan(\frac{5}{3}) \approx 59.04^\circ$
4. $-2 + 7i$ in quadrant II
   $\theta = \arctan(\frac{7}{-2}) + 180^\circ$
   $\approx -74.05^\circ + 180^\circ \approx 105.95^\circ$
5. $4 - 4i$ in quadrant IV
   $\theta = \arctan(\frac{-4}{4}) = -45^\circ$
   or: $\theta = -45^\circ + 360^\circ = 315^\circ$
6. $-2 - 2\sqrt{3}i$ in quadrant III
   $\theta = \arctan(\frac{-2\sqrt{3}}{-2}) + 180^\circ$
   $= 60^\circ + 180^\circ = 240^\circ$
7. $-4\sqrt{3} + 4i$ in quadrant II
   $\theta = \arctan(\frac{4}{-4\sqrt{3}}) + 180^\circ$
   $= -30^\circ + 180^\circ = 150^\circ$

Polar form of a complex number

Polar form and standard form of a complex number



Standard form: $\boxed{a+bi}$

Since $\cos(\theta) = \frac{a}{c}$ and $\sin(\theta) = \frac{b}{c}$, we get:

$a = c\cdot\cos(\theta)$ and $b = c\cdot\sin(\theta)$ and thus:

$$a+bi = c\cos(\theta) + c\sin(\theta)i = c(\cos(\theta)+i\sin(\theta))$$

Polar form: $\boxed{c\cdot(\cos(\theta)+i\sin(\theta))}$

Convert to polar form:

1. $-3-3i$

   $c = |-3-3i| = \sqrt{(-3)^2+(-3)^2}$
   $= \sqrt{18} = 3\sqrt{2}$

   $\theta = \arctan(\frac{-3}{-3}) + 180^\circ$
   $= 45^\circ + 180^\circ = 225^\circ$

   Therefore:

   $-3-3i = 3\sqrt{2}\cdot(\cos(225^\circ)+i\sin(225^\circ))$

2. $4\sqrt{3}-4i$

   $c = |4\sqrt{3}-4i| = \sqrt{(4\sqrt{3})^2+(-4)^2}$
   $= \sqrt{16\cdot 3+16} = \sqrt{64} = 8$

   $\theta = \arctan(\frac{-4}{4\sqrt{3}}) = -30^\circ$

   Therefore:

   $4-4\sqrt{3}i = 8\cdot(\cos(-30^\circ)+i\sin(-30^\circ))$

Converting polar form ↔ standard form - exercises

Convert to polar form:

3. $-7 + 7\sqrt{3}i$

   $c = |-7\sqrt{3}+7i| = \sqrt{(-7)^2 + (7\sqrt{3})^2}$

   $= \sqrt{49 + 49 \cdot 3} = \sqrt{196} = 14$

   $\theta = \arctan(\frac{7\sqrt{3}}{-7}) + 180^\circ$

   $= -60^\circ + 180^\circ = 120^\circ$

   Therefore:

   $-7 + 7\sqrt{3}i$

   $= 14 \cdot (\cos(120^\circ) + i\sin(120^\circ))$

Convert to standard form:

1. $4 \cdot (\cos(210^\circ) + i\sin(210^\circ))$

   $= 4 \cdot (\frac{-\sqrt{3}}{2} + i\frac{-1}{2})$

   $= 4 \cdot \frac{-\sqrt{3}}{2} + 4 \cdot \frac{-1}{2} \cdot i$

   $= -2\sqrt{3} - 2i$

Convert to standard form:

2. $6 \cdot (\cos(300^\circ) + i\sin(300^\circ))$

   $= 6 \cdot (\frac{1}{2} + i\frac{-\sqrt{3}}{2})$

   $= 6 \cdot \frac{1}{2} + 6 \cdot \frac{-\sqrt{3}}{2} \cdot i$

   $= 3 - 3\sqrt{3}i$

3. $7 \cdot (\cos(135^\circ) + i\sin(135^\circ))$

   $= 7 \cdot (\frac{-\sqrt{2}}{2} + i\frac{\sqrt{2}}{2})$

   $= \frac{-7\sqrt{2}}{2} + \frac{7\sqrt{2}}{2} \cdot i$

4. $5 \cdot (\cos(270^\circ) + i\sin(270^\circ))$

   $= 5 \cdot (0 + i(-1))$

   $= 0 - 5i$

   $= -5i$

Multiplying and dividing complex numbers in polar form

Multiplying and dividing complex numbers in polar form

Multiplication in polar form:

$$c_1(\cos\theta_1 + i\sin\theta_1)\cdot c_2(\cos\theta_2 + i\sin\theta_2) = c_1c_2\cdot(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2))$$

Division in polar form:

$$\frac{c_1(\cos\theta_1 + i\sin\theta_1)}{c_2(\cos\theta_2 + i\sin\theta_2)} = \frac{c_1}{c_2}\cdot(\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2))$$

Perform the multiplication or division:

1. $3(\cos 77^\circ + i\sin 77^\circ)\cdot 4(\cos 54^\circ + i\sin 54^\circ)$
   $= 12\cdot(\cos 131^\circ + i\sin 131^\circ)$
2. $6(\cos 99^\circ + i\sin 99^\circ)\cdot 10(\cos 183^\circ + i\sin 183^\circ)$
   $= 60\cdot(\cos 282^\circ + i\sin 282^\circ)$
3. $\frac{8(\cos 149^\circ + i\sin 149^\circ)}{6(\cos 36^\circ + i\sin 36^\circ)}$
   $= \frac{4}{3}\cdot(\cos 113^\circ + i\sin 113^\circ)$
4. $\frac{20(\cos 50^\circ + i\sin 50^\circ)}{4(\cos 70^\circ + i\sin 70^\circ)}$
   $= 5\cdot(\cos(-20^\circ) + i\sin(-20^\circ))$

Multiplying and dividing complex numbers in polar form - exercises

Multiply or divide and write your answer in standard form without approximation:

1. $2(\cos 35^\circ + i\sin 35^\circ)\cdot 7(\cos 100^\circ + i\sin 100^\circ)$
   $= 14\cdot(\cos 135^\circ + i\sin 135^\circ)$
   $= 14\cdot(\frac{-\sqrt{2}}{2} + i\frac{\sqrt{2}}{2})$
   $= -7\sqrt{2} + 7\sqrt{2}i$

2. $3(\cos 123^\circ + i\sin 123^\circ)\cdot 5(\cos 177^\circ + i\sin 177^\circ)$
   $= 15\cdot(\cos 300^\circ + i\sin 300^\circ)$
   $= 15\cdot(\frac{1}{2} + \frac{-\sqrt{3}}{2}i)$
   $= \frac{15}{2} - \frac{15\sqrt{3}}{2}i$

3. $8(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})\cdot 4(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$
   $= 32\cdot(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})$
   $= 32\cdot(0 + i\cdot 1)$
   $= 0 + 32i$
   $= 32i$

4. $\frac{3(\cos 117^\circ + i\sin 117^\circ)}{9(\cos 87^\circ + i\sin 87^\circ)}$
   $= \frac{1}{3}\cdot(\cos 30^\circ + i\sin 30^\circ)$
   $= \frac{1}{3}\cdot(\frac{\sqrt{3}}{2} + \frac{1}{2}i)$
   $= \frac{\sqrt{3}}{6} + \frac{1}{6}i$

5. $\frac{2(\cos 335^\circ + i\sin 335^\circ)}{2(\cos 110^\circ + i\sin 110^\circ)}$
   $= 1\cdot(\cos 225^\circ + i\sin 225^\circ)$
   $= -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$

6. $\frac{10(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3})}{25(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})}$
   $= \frac{2}{5}\cdot(\cos(\frac{5\pi}{3} - \frac{\pi}{6}) + i\sin(\frac{5\pi}{3} - \frac{\pi}{6}))$
   $= \frac{2}{5}\cdot(\cos(\frac{10\pi-\pi}{6}) + i\sin(\frac{10\pi-\pi}{6}))$
   $= \frac{2}{5}\cdot(\cos(\frac{9\pi}{6}) + i\sin(\frac{9\pi}{6}))$
   $= \frac{2}{5}\cdot(\cos(\frac{3\pi}{2}) + i\sin(\frac{3\pi}{2}))$
   $= \frac{2}{5}\cdot(0 + i(-1))$
   $= 0 - \frac{2}{5}i$