Vectors in the plane

Lesson #22

MAT 1375 Precalculus

New York City College of Technology CUNY

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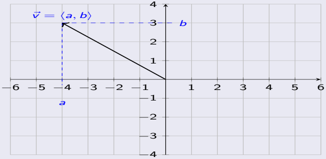
Vector in the plane



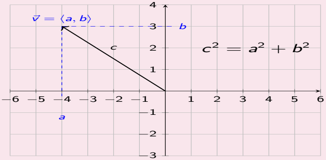
A **vector** is a geometric object that is given by a **direction** and **magnitude**.

For a vector in the plane, we write the vector in coordinates:

$$\vec{v} = \langle a, b \rangle \quad \text{or} \quad \vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}$$

Magnitude of a vector in the plane

The **magnitude** of $\vec{v} = \langle a, b \rangle$ is



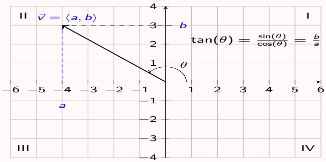
$$||\vec{v}|| = ||\langle a, b \rangle|| = \sqrt{a^2 + b^2}$$

Find the magnitude of the vector.

1. $\vec{v} = \langle 4, -2 \rangle$
   $\Rightarrow ||\langle 4, -2 \rangle|| = \sqrt{(4)^2 + (-2)^2}$
   $= \sqrt{16 + 4} = \sqrt{20} = \sqrt{4 \cdot 5} = 2\sqrt{5}$
2. $\vec{v} = \langle -3, -3 \rangle$
   $\Rightarrow ||\langle -3, -3 \rangle|| = \sqrt{(-3)^2 + (-3)^2}$
   $= \sqrt{9 + 9} = \sqrt{18} = \sqrt{9 \cdot 2} = 3\sqrt{2}$
3. $\vec{v} = \langle -5, 5\sqrt{3} \rangle$
   $\Rightarrow ||\langle -5, 5\sqrt{3} \rangle|| = \sqrt{(-5)^2 + (5\sqrt{3})^2}$
   $= \sqrt{25 + 25 \cdot 3} = \sqrt{100} = 10$

Direction angle of a vector in the plane

The **direction angle** of $\vec{v} = \langle a, b \rangle$ is



$\theta = \arctan(\frac{b}{a})$	if in I or IV
$\theta = \arctan(\frac{b}{a}) + 180^\circ$	if in II or III

Find the direction angle of the vector.

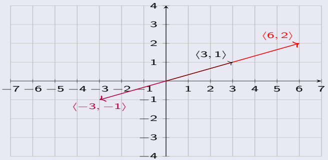
1. $\vec{v} = \langle 3, 2 \rangle$
   $\theta = \arctan(\frac{2}{3}) \approx 33.69^\circ$
2. $\vec{v} = \langle -3, -2 \rangle$ in quadrant III
   $\theta = \arctan(\frac{-2}{-3}) + 180^\circ$
   $\approx 33.69^\circ + 180^\circ \approx 213.69^\circ$

Find the magnitude and direction angle.

1. $\vec{v} = \langle 4, 3 \rangle$ in quadrant I
   $||\vec{v}|| = \sqrt{16 + 9} = \sqrt{25} = 5$
   $\theta = \arctan(\frac{3}{4}) \approx 36.87^\circ$
2. $\vec{v} = \langle -2, 2 \rangle$ in quadrant II
   $||\vec{v}|| = \sqrt{4 + 4} = \sqrt{8} = 2\sqrt{2}$
   $\theta = \arctan(\frac{2}{-2}) + 180^\circ = -45^\circ + 180^\circ = 135^\circ$
3. $\vec{v} = \langle -5, -5 \rangle$ in quadrant III
   $||\vec{v}|| = \sqrt{25 + 25} = \sqrt{50} = 5\sqrt{2}$
   $\theta = \arctan(\frac{-5}{-5}) + 180^\circ = 45^\circ + 180^\circ = 225^\circ$
4. $\vec{v} = \langle 7, -7\sqrt{3} \rangle$ in quadrant IV
   $||\vec{v}|| = \sqrt{49 + 49 \cdot 3} = \sqrt{196} = 14$
   $\theta = \arctan(\frac{-7\sqrt{3}}{7}) = -60^\circ$
5. $\vec{v} = \langle -\sqrt{3}, 1 \rangle$ in quadrant II
   $||\vec{v}|| = \sqrt{3 + 1} = \sqrt{4} = 2$
   $\theta = \arctan(\frac{1}{-\sqrt{3}}) + 180^\circ = -30^\circ + 180^\circ = 150^\circ$
6. $\vec{v} = \langle -10, -6 \rangle$ in quadrant III
   $||\vec{v}|| = \sqrt{100 + 36} = \sqrt{136} = 2\sqrt{34}$
   $\theta = \arctan(\frac{-6}{-10}) + 180^\circ \approx 30.96^\circ + 180^\circ \approx 210.96^\circ$

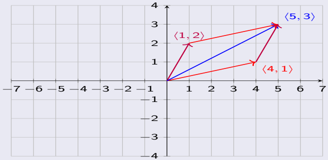
Scalar multiplication and vector addition

Scalar multiplication



$$r \cdot \langle a, b\rangle = \langle r \cdot a, r \cdot b\rangle$$

Vector addition



$$\langle a, b\rangle + \langle c, d\rangle = \langle a + c, b + d\rangle$$

Multiply the scalar with the vector.

1. $2 \cdot \langle 3, 1\rangle = \langle 6, 2\rangle$
2. $-1 \cdot \langle 3, 1\rangle = \langle -3, -1\rangle$
3. $5 \cdot \langle -7, 3\rangle = \langle -35, 15\rangle$
4. $\frac{1}{4} \cdot \langle 6, -7\rangle = \langle \frac{6}{4}, -\frac{7}{4}\rangle = \langle \frac{3}{2}, -\frac{7}{4}\rangle$
5. $-\frac{3}{8} \cdot \langle -\frac{5}{6}, \frac{11}{9}\rangle = \langle \frac{15}{48}, -\frac{33}{72}\rangle$
   $= \langle \frac{5}{16}, -\frac{11}{24}\rangle$

Add the vectors.

1. $\langle 4, 1\rangle + \langle 1, 2\rangle = \langle 5, 3\rangle$
2. $\langle 6, -2\rangle + \langle 7, 5\rangle = \langle 13, 3\rangle$
3. $\langle -2, -4\rangle + \langle -6, 4\rangle = \langle -8, 0\rangle$
4. $\langle 3, -1\rangle - \langle -5, 3\rangle = \langle 8, -4\rangle$

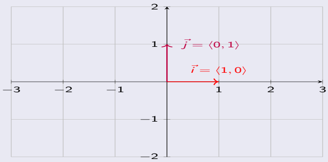
Scalar multiplication and vector addition - exercises

Evaluate the expression.

1. $\langle -8, -5\rangle - 2\cdot\langle -4, 9\rangle$
   $= \langle -8 - 2\cdot(-4), -5 - 2\cdot 9\rangle$
   $= \langle -8 + 8, -5 - 18\rangle$
   $= \langle 0, -23\rangle$
2. $(-3)\cdot\langle 2, 4\rangle + 5\cdot\langle 7, -3\rangle$
   $= \langle -6, -12\rangle + \langle 35, -15\rangle$
   $= \langle 29, -27\rangle$
3. $\vec{v} = \langle 6, -8\rangle,\ \vec{w} = \langle -4, -9\rangle$
   Find $4\cdot\vec{v} - 7\cdot\vec{w}$
   $= 4\cdot\langle 6, -8\rangle - 7\cdot\langle -4, -9\rangle$
   $= \langle 24, -32\rangle + \langle 28, 63\rangle$
   $= \langle 52, 31\rangle$
4. $-\frac{2}{9}\cdot\langle -\frac{5}{6}, \frac{8}{7}\rangle + \frac{1}{7}\cdot\langle \frac{11}{4}, -\frac{2}{3}\rangle$
   $= \langle \frac{5}{27}, -\frac{16}{63}\rangle + \langle \frac{11}{28}, -\frac{2}{21}\rangle$
   $= \langle \frac{5\cdot 28 + 11\cdot 27}{27\cdot 28}, \frac{-16-2\cdot 3}{63}\rangle$
   $= \langle \frac{437}{756}, -\frac{22}{63}\rangle$

Standard unit vectors

Denote by $\vec{i} = \langle 1, 0\rangle$ and $\vec{j} = \langle 0, 1\rangle$.



5. $2\vec{i} + (-7)\vec{j} + (-6)\vec{i} + 9\vec{j}$
   $= -4\vec{i} + 2\vec{j}$
   $= -4\cdot\langle 1, 0\rangle + 2\cdot\langle 0, 1\rangle$
   $= \langle -4, 2\rangle$
6. $-3\vec{i} - 2\vec{j} + 8\vec{i} - 4\vec{j}$
   $= 5\vec{i} - 6\vec{j}$
   $= \langle 5, -6\rangle$

Unit vectors

Definition

A unit vector $\vec{u}$ is a vector with magnitude 1.

$$||\vec{u}|| = 1$$

Recall: $||\langle a, b\rangle|| = \sqrt{a^2 + b^2}$

Rescaling any vector $\vec{v} = \langle a, b\rangle$ by its magnitude gives a unit vector $\vec{u}$ in the same direction as $\vec{v}$.

$$\vec{u} = \frac{1}{||\vec{v}||} \cdot \vec{v}$$

Find the unit vector in the same direction as the given vectors.

1. $\vec{v} = \langle 8, 6\rangle$
   $\Rightarrow ||\vec{v}|| = \sqrt{8^2 + 6^2}$
   $= \sqrt{64 + 36} = \sqrt{100} = 10$
   $\Rightarrow \vec{u} = \frac{1}{10} \cdot \langle 8, 6\rangle = \langle \frac{4}{5}, \frac{3}{5}\rangle$
2. $\vec{v} = \langle -5, 12\rangle$
   $\Rightarrow ||\vec{v}|| = \sqrt{(-5)^2 + 12^2}$
   $= \sqrt{25 + 144} = \sqrt{169} = 13$
   $\Rightarrow \vec{u} = \frac{1}{13} \cdot \langle -5, 12\rangle = \langle \frac{-5}{13}, \frac{12}{13}\rangle$
3. $\vec{v} = \langle 4, -2\rangle$
   $\Rightarrow ||\vec{v}|| = \sqrt{4^2 + (-2)^2}$
   $= \sqrt{16 + 4} = \sqrt{20} = 2\sqrt{5}$
   $\Rightarrow \vec{u} = \frac{1}{2\sqrt{5}} \cdot \langle 4, -2\rangle$
   $= \langle \frac{4}{2\sqrt{5}}, \frac{-2}{2\sqrt{5}}\rangle = \langle \frac{2}{\sqrt{5}}, \frac{-1}{\sqrt{5}}\rangle$
   $= \langle \frac{2\sqrt{5}}{5}, \frac{-\sqrt{5}}{5}\rangle$