Trigonometric identities

Lesson #21

MAT 1375 Precalculus

New York City College of Technology CUNY

Reciprocal identities and quotient identities

Reciprocal and quotient identities

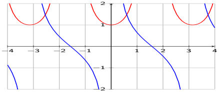
$$\tan(x) = \frac{\sin(x)}{\cos(x)} \qquad \csc(x) = \frac{1}{\sin(x)} \qquad \sec(x) = \frac{1}{\cos(x)}$$

$$\cot(x) = \frac{1}{\tan(x)} = \frac{\cos(x)}{\sin(x)} \qquad \sin(x) = \frac{1}{\csc(x)} \qquad \cos(x) = \frac{1}{\sec(x)}$$

1. Simplify $\sin(x) \cdot \cot(x)$.
   Solution:
   $\sin(x) \cdot \cot(x) = \sin(x) \cdot \frac{\cos(x)}{\sin(x)} = \cos(x)$

2. Simplify $\csc(x) \cdot \tan(x)$.
   Solution:
   $$\csc(x) \cdot \tan(x) = \frac{1}{\sin(x)} \cdot \frac{\sin(x)}{\cos(x)} = \frac{1}{\cos(x)} = \sec(x)$$

3. True or false? $\frac{\sec(x)}{\cos(x)} = \cot(x)$
   False, their graphs are different:



$\frac{\sec(x)}{\cos(x)}$

$\cot(x)$

4. True or false? $\frac{\sec(x)}{\csc(x)} = \tan(x)$
   True (same graphs!) Proof of identity:
   $$\frac{\sec(x)}{\csc(x)} = \frac{\frac{1}{\cos(x)}}{\frac{1}{\sin(x)}} = \frac{1}{\cos(x)} \cdot \frac{\sin(x)}{1} = \frac{\sin(x)}{\cos(x)} = \tan(x)$$

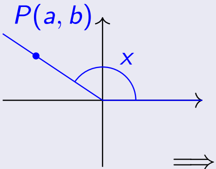
Pythagorean identities

$$\boxed{\sin^2(x) + \cos^2(x) = 1}$$

$$\boxed{\sec^2(x) = 1 + \tan^2(x)}$$

$$\csc^2(x) = 1 + \cot^2(x)$$

Proof:



$$\sin(x) = b$$

$$\cos(x) = a$$

P on unit circle

$$\implies \quad a^2 + b^2 = 1$$

Therefore: $\sin^2(x) + \cos^2(x) = b^2 + a^2 = 1$

$$1 + \tan^2(x) = 1 + (\tfrac{b}{a})^2 = \tfrac{a^2+b^2}{a^2} = \tfrac{1}{a^2} = \sec^2(x)$$

$$1 + \cot^2(x) = 1 + (\tfrac{a}{b})^2 = \tfrac{b^2+a^2}{b^2} = \tfrac{1}{b^2} = \csc^2(x)$$

1. Simplify: $\dfrac{1 - \sin^2(x)}{\cos(x)}$

   Solution:
   $$\frac{1 - \sin^2(x)}{\cos(x)} = \frac{\cos^2(x)}{\cos(x)} = \cos(x)$$

2. Simplify: $\dfrac{1 - \sec^2(x)}{\tan(x)}$

   Solution:
   $$\frac{1 - \sec^2(x)}{\tan(x)} = \frac{-\tan^2(x)}{\tan(x)} = -\tan(x)$$

3. Simplify: $\dfrac{\cot^2(x) - \csc^2(x)}{\cos(x)}$

   Solution:
   $$\frac{\cot^2(x) - \csc^2(x)}{\cos(x)} = \frac{-1}{\cos(x)} = -\sec(x)$$

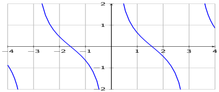
Pythagorean identities - exercises

4. Simplify: $(\cos(x)+1)\cdot(\cos(x)-1)$
   Solution:
   Recall: $\boxed{(a+b)(a-b)=a^2-b^2}$
   $$\Longrightarrow (\cos(x)+1)\cdot(\cos(x)-1)$$
   $$=\cos^2(x)-1=-\sin^2(x)$$

5. Simplify:
   $(\tan(x)-\sec(x))\cdot(\tan(x)+\sec(x))$
   Solution:
   $$\Longrightarrow (\tan(x)-\sec(x))\cdot(\tan(x)+\sec(x))$$
   $$=\tan^2(x)-\sec^2(x)=-1$$

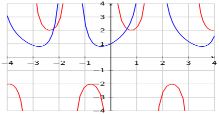
6. Simplify: $(1-\csc(x))\cdot(1+\csc(x))$
   Solution:
   $$\Longrightarrow (1-\csc(x))\cdot(1+\csc(x))$$
   $$=1-\csc^2(x)=-\cot^2(x)$$

7. True or false?
   $$\frac{\sec(x)}{\sin(x)}-\frac{\sin(x)}{\cos(x)}=\cot(x)$$
   True. Both graphs coincide:



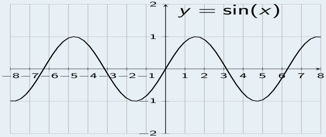
Proof of the identity:
$$\frac{\sec(x)}{\sin(x)}-\frac{\sin(x)}{\cos(x)}=\frac{\sec(x)\cos(x)-\sin(x)\sin(x)}{\sin(x)\cos(x)}$$
$$=\frac{1-\sin^2(x)}{\sin(x)\cos(x)}=\frac{\cos^2(x)}{\sin(x)\cos(x)}$$
$$=\frac{\cos(x)}{\sin(x)}=\cot(x)$$

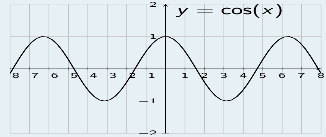
8. True or false?
   $$\frac{\csc(x)}{\cos(x)}=\sin(x)+\frac{\sec(x)}{\cos(x)}$$
   False, since graphs do not coincide:



Even, odd trigonometric functions

$\sin(-x) = -\sin(x)$	odd
$\cos(-x) = \cos(x)$	even
$\tan(-x) = -\tan(x)$	odd
$\csc(-x) = -\csc(x)$	odd
$\sec(-x) = \sec(x)$	even
$\cot(-x) = -\cot(x)$	odd

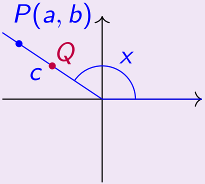




General point P on the terminal side

$P(a, b)$ is a point on the terminal side of x

$$a^2 + b^2 = c^2 \quad \Longrightarrow \quad c = \sqrt{a^2 + b^2}$$



$Q\left(\frac{a}{c}, \frac{b}{c}\right)$ is on the unit circle

$$\sin(x) = \frac{b}{c} \qquad \csc(x) = \frac{c}{b}$$

$$\cos(x) = \frac{a}{c} \qquad \sec(x) = \frac{c}{a}$$

$$\tan(x) = \frac{b}{a} \qquad \cot(x) = \frac{a}{b}$$

Adding $\pm\frac{\pi}{2}$ or $\pm\pi$; addition, subtraction, half-, double-angle formulas

Identities from adding $\pm\frac{\pi}{2}$ or $\pm\pi$

$$\begin{aligned}
\sin(\pi - x) &= \sin(x) \\
\cos(\pi - x) &= -\cos(x) \\
\sin(x + \frac{\pi}{2}) &= \cos x \\
\cos(x + \frac{\pi}{2}) &= -\sin(x) \\
\sin(x - \frac{\pi}{2}) &= -\cos x \\
\cos(x - \frac{\pi}{2}) &= \sin(x)
\end{aligned}$$

Sum and difference formulas

$$\begin{aligned}
\sin(\alpha + \beta) &= \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta) \\
\sin(\alpha - \beta) &= \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta) \\
\cos(\alpha + \beta) &= \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) \\
\cos(\alpha - \beta) &= \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta) \\
\tan(\alpha + \beta) &= \frac{\tan(\alpha) + \tan(\beta)}{1 - \tan(\alpha)\tan(\beta)} \\
\tan(\alpha - \beta) &= \frac{\tan(\alpha) - \tan(\beta)}{1 + \tan(\alpha)\tan(\beta)}
\end{aligned}$$

Half-angle formulas

$$\begin{aligned}
\sin(\frac{\alpha}{2}) &= \pm\sqrt{\frac{1-\cos(\alpha)}{2}} \\
\cos(\frac{\alpha}{2}) &= \pm\sqrt{\frac{1+\cos(\alpha)}{2}} \\
\tan(\frac{\alpha}{2}) &= \pm\sqrt{\frac{1-\cos(\alpha)}{1+\cos(\alpha)}}
\end{aligned}$$

Double-angle formulas

$$\begin{aligned}
\sin(2\alpha) &= 2\sin(\alpha)\cos(\alpha) \\
\cos(2\alpha) &= \cos^2(\alpha) - \sin^2(\alpha) \\
&= 1 - 2\sin^2(\alpha) = 2\cos^2(\alpha) - 1 \\
\tan(2\alpha) &= \frac{2\tan(\alpha)}{1 - \tan^2(\alpha)}
\end{aligned}$$