

Trigonometric equations

Lesson #20

MAT 1375 Precalculus

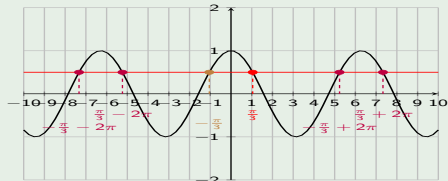
New York City College of Technology CUNY



Solving equations $\cos(x) = c$

Example

Solve for x : $\cos(x) = \frac{1}{2}$



One solution comes from

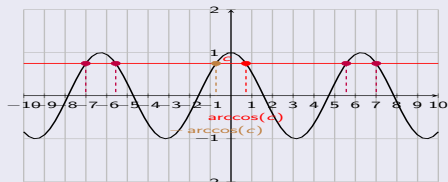
$$x = \arccos\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

A second solution is: $x = -\frac{\pi}{3}$.

All other solutions come from adding or subtracting 2π s. The general solution is:

$$\begin{cases} x = \frac{\pi}{3} + n \cdot 2\pi, & \text{for } n \in \mathbb{Z} \\ x = -\frac{\pi}{3} + n \cdot 2\pi, & \text{for } n \in \mathbb{Z} \end{cases}$$

Solving $\cos(x) = c$



One solution comes from

$$x = \arccos(c)$$

A second solution is: $x = -\arccos(c)$.

All other solutions come from adding or subtracting multiple of 2π s:

$$\begin{cases} x = \arccos(c) + n \cdot 2\pi, & \text{for } n \in \mathbb{Z} \\ x = -\arccos(c) + n \cdot 2\pi, & \text{for } n \in \mathbb{Z} \end{cases}$$

Solving equations with \cos - exercises

Solve $\cos(x) = c$

Solution:

$$x = \arccos(c) + n \cdot 2\pi, \quad \text{for } n \in \mathbb{Z}$$

$$x = -\arccos(c) + n \cdot 2\pi, \quad \text{for } n \in \mathbb{Z}$$

① Solve for x : $\cos(x) = \frac{\sqrt{2}}{2}$

Solution:

$$\arccos\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$$

$$-\arccos\left(\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$$

$$\Rightarrow x = \frac{\pi}{4} + n2\pi, \text{ for } n \in \mathbb{Z}$$

$$\text{or } x = -\frac{\pi}{4} + n2\pi, \text{ for } n \in \mathbb{Z}$$

② Solve for x : $\cos(x) = \frac{\sqrt{3}}{2}$

Solution:

$$\arccos\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$$

$$-\arccos\left(\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{6}$$

$$\Rightarrow x = \frac{\pi}{6} + 2n\pi, \text{ for } n \in \mathbb{Z}$$

$$\text{or } x = -\frac{\pi}{6} + 2n\pi, \text{ for } n \in \mathbb{Z}$$

③ Solve for x : $2\cos(x) + 1 = 0$

Solution:

$$\Rightarrow 2\cos(x) = -1 \Rightarrow \cos(x) = -\frac{1}{2}$$

$$\arccos\left(-\frac{1}{2}\right) = 120^\circ = \frac{2\pi}{3}$$

$$-\arccos\left(-\frac{1}{2}\right) = -\frac{2\pi}{3}$$

$$\Rightarrow x = \frac{2\pi}{3} + 2n\pi, \text{ for } n \in \mathbb{Z}$$

$$\text{or } x = -\frac{2\pi}{3} + 2n\pi, \text{ for } n \in \mathbb{Z}$$

④ Solve for x : $\cos^2(x) + \cos(x) = 0$

Solution:

$$\Rightarrow \cos(x) \cdot (\cos(x) + 1) = 0$$

$$\Rightarrow \cos(x) = 0 \text{ or } \cos(x) = -1$$

$$\pm \arccos(0) = \pm \frac{\pi}{2}$$

$$\pm \arccos(-1) = \pm \pi$$

$$\Rightarrow x = \frac{\pi}{2} + 2n\pi, \text{ for } n \in \mathbb{Z}$$

$$\text{or } x = -\frac{\pi}{2} + 2n\pi, \text{ for } n \in \mathbb{Z}$$

$$\text{or } x = \pi + 2n\pi, \text{ for } n \in \mathbb{Z}$$

$$\text{or } x = -\pi + 2n\pi, \text{ for } n \in \mathbb{Z}$$

Solving equations with \cos - exercises

Solve $\cos(x) = c$

Solution:

$$x = \arccos(c) + n \cdot 2\pi, \quad \text{for } n \in \mathbb{Z}$$

$$x = -\arccos(c) + n \cdot 2\pi, \quad \text{for } n \in \mathbb{Z}$$

- 5 Solve for x :

$$\cos^2(x) - 2\cos(x) - 3 = 0$$

Solution:

Substitute $u = \cos(x)$

$$\Rightarrow u^2 - 2u - 3 = 0$$

$$\Rightarrow (u - 3)(u + 1) = 0$$

$$\Rightarrow u = 3 \text{ or } u = -1$$

$$\Rightarrow \cos(x) = 3 \quad \text{or} \quad \cos(x) = -1$$

Note: $\cos(x) = 3$ has no solutions, since the range of \cos is $[-1, 1]$.

$$\arccos(-1) = \pi$$

$$-\arccos(-1) = -\pi$$

$$\Rightarrow x = \pi + 2n\pi, \text{ for } n \in \mathbb{Z}$$

$$\text{or } x = -\pi + 2n\pi, \text{ for } n \in \mathbb{Z}$$

- 6 Solve for x : $4\cos^2(x) - 3 = 0$

Solution:

Substitute $u = \cos(x)$

$$\Rightarrow 4u^2 - 3 = 0 \Rightarrow 4u^2 = 3$$

$$\Rightarrow u^2 = \frac{3}{4} \Rightarrow u = \pm \sqrt{\frac{3}{4}} = \pm \frac{\sqrt{3}}{2}$$

$$\Rightarrow \cos(x) = \frac{\sqrt{3}}{2} \quad \text{or} \quad \cos(x) = -\frac{\sqrt{3}}{2}$$

$$\arccos\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$$

$$-\arccos\left(\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{6}$$

$$\arccos\left(-\frac{\sqrt{3}}{2}\right) = 150^\circ = \frac{5\pi}{6}$$

$$-\arccos\left(-\frac{\sqrt{3}}{2}\right) = -\frac{5\pi}{6}$$

$$\Rightarrow x = \frac{\pi}{6} + 2n\pi, \text{ for } n \in \mathbb{Z}$$

$$\text{or } x = -\frac{\pi}{6} + 2n\pi, \text{ for } n \in \mathbb{Z}$$

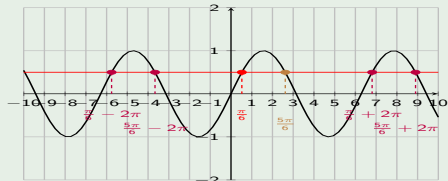
$$\text{or } x = \frac{5\pi}{6} + 2n\pi, \text{ for } n \in \mathbb{Z}$$

$$\text{or } x = -\frac{5\pi}{6} + 2n\pi, \text{ for } n \in \mathbb{Z}$$

Solving equations $\sin(x) = c$

Example

Solve for x : $\sin(x) = \frac{1}{2}$



One solution comes from

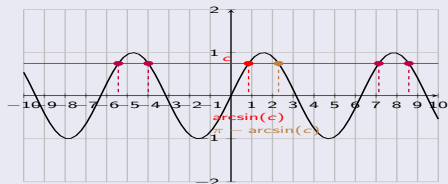
$$x = \arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

A second solution is: $x = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$.

All other solutions come from adding or subtracting 2π s. The general solution is:

$$\begin{cases} x = \frac{\pi}{6} + n \cdot 2\pi, & \text{for } n \in \mathbb{Z} \\ x = \frac{5\pi}{6} + n \cdot 2\pi, & \text{for } n \in \mathbb{Z} \end{cases}$$

Solving $\sin(x) = c$



One solution comes from

$$x = \arcsin(c)$$

A second solution is: $x = \pi - \arcsin(c)$.

All other solutions come from adding or subtracting multiple of 2π s:

$$\begin{cases} x = \arcsin(c) + n \cdot 2\pi, & \text{for } n \in \mathbb{Z} \\ x = \pi - \arcsin(c) + n \cdot 2\pi, & \text{for } n \in \mathbb{Z} \end{cases}$$

Solving equations with sin - exercises

Solve $\sin(x) = c$

Solution:

$$x = \arcsin(c) + n \cdot 2\pi, \quad \text{for } n \in \mathbb{Z}$$

$$x = \pi - \arcsin(c) + n \cdot 2\pi, \quad \text{for } n \in \mathbb{Z}$$

1 Solve for x : $\sin(x) = -\frac{\sqrt{2}}{2}$

Solution:

$$\arcsin\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$$

$$\pi - \arcsin\left(\frac{\sqrt{2}}{2}\right) = \pi - \left(-\frac{\pi}{4}\right) = \pi + \frac{\pi}{4} = \frac{5\pi}{4}$$

$$\Rightarrow x = -\frac{\pi}{4} + 2n\pi, \quad \text{for } n \in \mathbb{Z}$$

$$\text{or } x = \frac{5\pi}{4} + 2n\pi, \quad \text{for } n \in \mathbb{Z}$$

2 Solve for x : $\sin(x) = -\frac{1}{2}$

Solution:

$$\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$$

$$\pi - \arcsin\left(\frac{1}{2}\right) = \pi + \frac{\pi}{6} = \frac{7\pi}{6}$$

$$\Rightarrow x = -\frac{\pi}{6} + 2n\pi, \quad \text{for } n \in \mathbb{Z}$$

$$\text{or } x = \frac{7\pi}{6} + 2n\pi, \quad \text{for } n \in \mathbb{Z}$$

3 Solve for x : $2\sin(x) - \sqrt{3} = 0$

Solution:

$$\Rightarrow 2\sin(x) = \sqrt{3} \Rightarrow \sin(x) = \frac{\sqrt{3}}{2}$$

$$\arcsin\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$$

$$\pi - \arcsin\left(\frac{\sqrt{3}}{2}\right) = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

$$\Rightarrow x = \frac{\pi}{3} + 2n\pi, \quad \text{for } n \in \mathbb{Z}$$

$$\text{or } x = \frac{2\pi}{3} + 2n\pi, \quad \text{for } n \in \mathbb{Z}$$

4 Solve for x : $2\sin^2(x) - 1 = 0$

Solution:

$$\text{Substitute } u = \sin(x) \Rightarrow 2u^2 - 1 = 0$$

$$\Rightarrow 2u^2 = 1 \Rightarrow u^2 = \frac{1}{2} \Rightarrow u = \pm\sqrt{\frac{1}{2}} = \pm\frac{\sqrt{2}}{2}$$

$$\Rightarrow \sin(x) = \frac{\sqrt{2}}{2} \quad \text{or} \quad \sin(x) = -\frac{\sqrt{2}}{2}$$

$$\arcsin\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$$

$$\pi - \arcsin\left(\frac{\sqrt{2}}{2}\right) = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

$$\arcsin\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$$

$$\pi - \arcsin\left(-\frac{\sqrt{2}}{2}\right) = \pi - \left(-\frac{\pi}{4}\right) = \frac{5\pi}{4}$$

$$\Rightarrow x = \frac{\pi}{4} + 2n\pi, \quad \text{for } n \in \mathbb{Z}$$

$$\text{or } x = \frac{3\pi}{4} + 2n\pi, \quad \text{for } n \in \mathbb{Z}$$

$$\text{or } x = -\frac{\pi}{4} + 2n\pi, \quad \text{for } n \in \mathbb{Z}$$

$$\text{or } x = \frac{5\pi}{4} + 2n\pi, \quad \text{for } n \in \mathbb{Z}$$

Solving equations with sin, cos, and tan - exercises

Summary for solving trigonometric equations

$\sin(x) = c$	$\cos(x) = c$	$\tan(x) = c$	
$x = \arcsin(c) + 2n\pi$	$x = \arccos(c) + 2n\pi$		for $n \in \mathbb{Z}$
$x = \pi - \arcsin(c) + 2n\pi$	$x = -\arccos(c) + 2n\pi$	$x = \arctan(c) + n\pi$	

① Solve: $\tan^2(x) + \tan(x) = 0$

Solution:

Substitute $u = \tan(x)$

$$\Rightarrow u^2 + u = 0 \Rightarrow u(u + 1) = 0$$

$$\Rightarrow u = 0 \quad \text{or} \quad u = -1$$

$$\Rightarrow \tan(x) = 0 \quad \text{or} \quad \tan(x) = -1$$

$$\arctan(0) = 0$$

$$\arctan(-1) = -\frac{\pi}{4}$$

$$\Rightarrow x = 0 + n\pi, \text{ for } n \in \mathbb{Z}$$

$$\Rightarrow x = -\frac{\pi}{4} + n\pi, \text{ for } n \in \mathbb{Z}$$

② Solve: $2\sin^2(x) + \sqrt{3}\sin(x) = 0$

Solution:

Substitute $u = \sin(x)$

$$\Rightarrow 2u^2 + \sqrt{3}u = 0 \Rightarrow u(2u + \sqrt{3}) = 0$$

$$\Rightarrow u = 0 \quad \text{or} \quad u = -\frac{\sqrt{3}}{2}$$

$$\Rightarrow \sin(x) = 0 \quad \text{or} \quad \sin(x) = -\frac{\sqrt{3}}{2}$$

$$\arcsin(0) = 0$$

$$\pi - \arcsin(0) = \pi - 0 = \pi$$

$$\arcsin\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$$

$$\pi - \arcsin\left(-\frac{\sqrt{3}}{2}\right) = \pi - \left(-\frac{\pi}{3}\right) = \frac{4\pi}{3}$$

$$\Rightarrow x = 0 + 2n\pi, \text{ for } n \in \mathbb{Z}$$

$$\Rightarrow x = \pi + 2n\pi, \text{ for } n \in \mathbb{Z}$$

$$\Rightarrow x = -\frac{\pi}{3} + 2n\pi, \text{ for } n \in \mathbb{Z}$$

$$\Rightarrow x = \frac{4\pi}{3} + 2n\pi, \text{ for } n \in \mathbb{Z}$$

Solving equations with sin, cos, and tan - exercises

Summary for solving trigonometric equations

$\sin(x) = c$	$\cos(x) = c$	$\tan(x) = c$	for $n \in \mathbb{Z}$
$x = \arcsin(c) + 2n\pi$ $x = \pi - \arcsin(c) + 2n\pi$	$x = \arccos(c) + 2n\pi$ $x = -\arccos(c) + 2n\pi$	$x = \arctan(c) + n\pi$	

③ Solve: $\cos(x)\tan(x) + \tan(x) = 0$

Solution:

$$\Rightarrow \tan(x) \cdot (\cos(x) + 1) = 0$$

$$\Rightarrow \tan(x) = 0 \quad \text{or} \quad \cos(x) = -1$$

$$\arctan(0) = 0$$

$$\arccos(-1) = \pi$$

$$-\arccos(-1) = -\pi$$

$$\Rightarrow x = 0 + n\pi, \text{ for } n \in \mathbb{Z}$$

$$\Rightarrow x = \pi + 2n\pi, \text{ for } n \in \mathbb{Z}$$

$$\Rightarrow x = -\pi + 2n\pi, \text{ for } n \in \mathbb{Z}$$

④ Solve: $\cos(x) - 2\cos(x)\sin(x) = 0$

Solution:

$$\Rightarrow \cos(x) \cdot (1 - 2\sin(x)) = 0$$

$$\Rightarrow \cos(x) = 0 \quad \text{or} \quad 1 - 2\sin(x) = 0$$

$$\Rightarrow \cos(x) = 0 \quad \text{or} \quad 2\sin(x) = 1$$

$$\Rightarrow \cos(x) = 0 \quad \text{or} \quad \sin(x) = \frac{1}{2}$$

$$\arccos(0) = \frac{\pi}{2}$$

$$-\arccos(0) = -\frac{\pi}{2}$$

$$\arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

$$\pi - \arcsin\left(\frac{1}{2}\right) = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$$

$$\Rightarrow x = \frac{\pi}{2} + 2n\pi, \text{ for } n \in \mathbb{Z}$$

$$\Rightarrow x = -\frac{\pi}{2} + 2n\pi, \text{ for } n \in \mathbb{Z}$$

$$\Rightarrow x = \frac{\pi}{6} + 2n\pi, \text{ for } n \in \mathbb{Z}$$

$$\Rightarrow x = \frac{5\pi}{6} + 2n\pi, \text{ for } n \in \mathbb{Z}$$

