Inverse trigonometric functions

Lesson #19

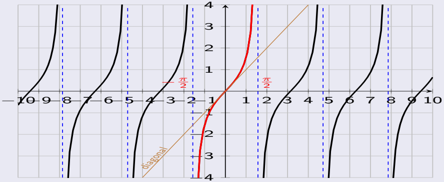
MAT 1375 Precalculus

New York City College of Technology CUNY

The inverse function of $y = \tan(x)$

Graph $y = \tan(x)$.

x	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
y	undef	$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undef



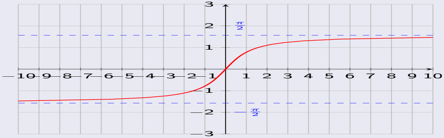
$y = \tan(x)$ is one-to-one when restricted to the domain $D = (-\frac{\pi}{2}, \frac{\pi}{2})$ with range $R = \mathbb{R}$.

Inverse tangent function

Define the inverse tangent function as:

$$x = \tan(y) \Leftrightarrow y = \tan^{-1}(x) = \arctan(x)$$

Domain: $D = \mathbb{R}$, Range $R = (-\frac{\pi}{2}, \frac{\pi}{2})$



Note: arctan is an odd function:

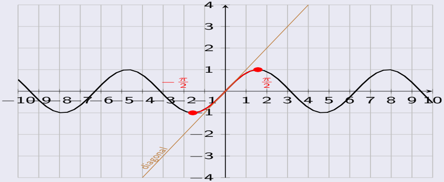
$$\arctan(-x) = -\arctan(x)$$

1. $\arctan(\sqrt{3}) =$
   $= \frac{\pi}{3} = 60^\circ$
2. $\arctan(-\frac{\sqrt{3}}{3}) =$
   $= -\frac{\pi}{6} = -30^\circ$
3. $\arctan(-1) =$
   $= -\frac{\pi}{4} = -45^\circ$
4. $\arctan(2 + \sqrt{3}) =$
   (use a calculator!)
   $= 75^\circ = 75 \cdot \frac{\pi}{180} = \frac{5\pi}{12}$
5. $\arctan(\sqrt{3} - 2) =$
   $= -15^\circ$
   $= -15 \cdot \frac{\pi}{180} = -\frac{\pi}{12}$
6. $\arctan(2) = ?$
   $\arctan(2) \approx 63.43^\circ$
   $\approx 63.43 \cdot \frac{\pi}{180} \approx 1.107$
7. $\arctan(-5) = ?$
   $\arctan(-5) \approx -78.69^\circ$
   ≈ -1.373

The inverse function of $y = \sin(x)$

Graph $y = \sin(x)$.

x	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
y	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1



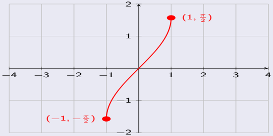
$y = \sin(x)$ is one-to-one when restricted to the domain $D = [-\frac{\pi}{2}, \frac{\pi}{2}]$ with range $R = [-1, 1]$.

Inverse sine function

Define the inverse sine function as:

$$x = \sin(y) \Leftrightarrow y = \sin^{-1}(x) = \arcsin(x)$$

Domain: $D = [-1, 1]$, Range $R = [-\frac{\pi}{2}, \frac{\pi}{2}]$



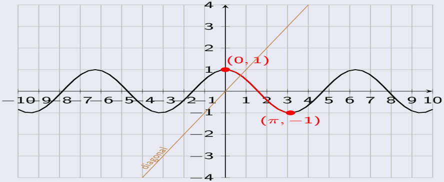
Note: arcsin is an odd function:

$$\arcsin(-x) = -\arcsin(x)$$

1. $\arcsin(\frac{\sqrt{2}}{2}) =$
   $= \frac{\pi}{4} = 45^\circ$
2. $\arcsin(-\frac{\sqrt{3}}{2}) =$
   $= -\frac{\pi}{3} = -60^\circ$
3. $\arcsin(1) =$
   $= \frac{\pi}{2} = 90^\circ$
4. $\arcsin(\frac{\sqrt{6}-\sqrt{2}}{4}) =$
   (use a calculator!)
   $= 15^\circ = 15 \cdot \frac{\pi}{180} = \frac{\pi}{12}$
5. $\arcsin(\frac{\sqrt{2-\sqrt{2}}}{2}) =$
   $= 22.5^\circ = 22.5 \cdot \frac{\pi}{180} = \frac{\pi}{8}$
6. $\arcsin(-0.3) = ?$
   $\arcsin(-0.3) \approx -17.46^\circ$
   ≈ -0.3047
7. $\arcsin(7) = ?$
   $\arcsin(7)$ is undefined, since the domain of arcsin is $D = [-1, 1]$.

Graph $y = \cos(x)$.

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
y	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1



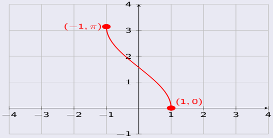
$y = \cos(x)$ is one-to-one when restricted to the domain $D = [0, \pi]$ with range $R = [-1, 1]$.

Inverse cosine function

Define the inverse cosine function as:

$$x = \cos(y) \Leftrightarrow y = \cos^{-1}(x) = \arccos(x)$$

Domain: $D = [-1, 1]$, Range $R = [0, \pi]$



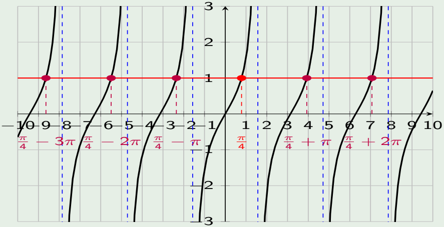
Note: arccos is neither even nor odd!
But: $\arccos(-x) = \pi - \arccos(x)$

1. $\arccos(\frac{1}{2}) =$
   $= \frac{\pi}{3} = 60^\circ$
2. $\arccos(-\frac{\sqrt{2}}{2}) =$
   $= \frac{3\pi}{4} = 135^\circ$
3. $\arccos(-\frac{1}{2}) =$
   $= \frac{2\pi}{3} = 120^\circ$
4. $\arccos(\frac{\sqrt{6}-\sqrt{2}}{4}) =$
   (use a calculator!)
   $= 75^\circ = 75 \cdot \frac{\pi}{180} = \frac{5\pi}{12}$
5. $\arccos(-\frac{1+\sqrt{3}}{2\sqrt{2}}) =$
   $= 165^\circ = 165 \cdot \frac{\pi}{180} = \frac{11\pi}{12}$
6. $\arccos(0.55) = ?$
   $\arccos(0.55) \approx 56.63^\circ$
   ≈ 0.988
7. $\arccos(-2.3) = ?$
   $\arccos(-2.3)$ is undefined, since the domain of arccos is $D = [-1, 1]$.

Solving equations $\tan(x) = c$

Example

Solve for x: $\tan(x) = 1$



One solution comes from

$$x = \arctan(1) = \frac{\pi}{4}$$

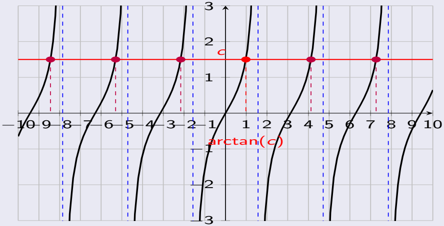
All other solutions come from adding or subtracting πs:

$$\ldots, \frac{\pi}{4} - 2\pi, \frac{\pi}{4} - \pi, \frac{\pi}{4}, \frac{\pi}{4} + \pi, \frac{\pi}{4} + 2\pi, \ldots$$

General solution:

$$x = \frac{\pi}{4} + n \cdot \pi, \quad \text{for } n \in \mathbb{Z}$$

Solving $\tan(x) = c$



One solution comes from

$$x = \arctan(c)$$

All other solutions come from adding or subtracting multiple of πs:

$$\boxed{x = \arctan(c) + n \cdot \pi} \quad \text{for } n \in \mathbb{Z}$$

Solving equations with tan - exercises

Solve $\tan(x) = c$

Solution: $\boxed{x = \arctan(c) + n \cdot \pi}$ for $n \in \mathbb{Z}$

1. Solve for x: $\tan(x) = -\sqrt{3}$
   Solution:
   $\arctan(-\sqrt{3}) = -\frac{\pi}{3}$
   $\Rightarrow x = -\frac{\pi}{3} + n\pi$, for $n \in \mathbb{Z}$
2. Solve for x: $\tan(x) = \frac{\sqrt{3}}{3}$
   Solution:
   $\arctan(\frac{\sqrt{3}}{3}) = \frac{\pi}{6}$
   $\Rightarrow x = \frac{\pi}{6} + n\pi$, for $n \in \mathbb{Z}$
3. Solve for x: $\tan(x) = 0$
   Solution:
   $\arctan(0) = 0$
   $\Rightarrow x = 0 + n\pi$, for $n \in \mathbb{Z}$
4. Solve for x: $\tan(x) + 1 = 0$
   Solution:
   $\Rightarrow \tan(x) = -1$
   $\arctan(-1) = -\frac{\pi}{4}$
   $\Rightarrow x = -\frac{\pi}{4} + n\pi$, for $n \in \mathbb{Z}$
5. Solve for x: $\tan(x) - \sqrt{3} = 0$
   Solution:
   $\Rightarrow \tan(x) = \sqrt{3}$
   $\arctan(\sqrt{3}) = \frac{\pi}{3}$
   $\Rightarrow x = \frac{\pi}{3} + n\pi$, for $n \in \mathbb{Z}$
6. Solve for x: $\tan^2(x) - \tan(x) = 0$
   Solution:
   $\Rightarrow \tan(x) \cdot (\tan(x) - 1) = 0$
   $\Rightarrow \tan(x) = 0$ or $\tan(x) = 1$
   $\arctan(0) = 0$
   $\arctan(1) = \frac{\pi}{4}$
   $\Rightarrow x = 0 + n\pi$, for $n \in \mathbb{Z}$
   or $x = \frac{\pi}{4} + n\pi$, for $n \in \mathbb{Z}$

Solving equations with tan - exercises

Solve $\tan(x) = c$

Solution: $\boxed{x = \arctan(c) + n \cdot \pi}$ for $n \in \mathbb{Z}$

7. Solve for x: $\tan^2(x) + \sqrt{3}\tan(x) = 0$
   Solution:
   Substitute $u = \tan(x)$
   $\Rightarrow u^2 + \sqrt{3}u = 0 \quad \Rightarrow u(u + \sqrt{3}) = 0$
   $\Rightarrow u = 0$ or $u = -\sqrt{3}$
   $\Rightarrow \tan(x) = 0$ or $\tan(x) = -\sqrt{3}$
   $\arctan(0) = 0$
   $\arctan(-\sqrt{3}) = -\frac{\pi}{3}$
   $\Rightarrow x = 0 + n\pi$, for $n \in \mathbb{Z}$
   or $x = -\frac{\pi}{3} + n\pi$, for $n \in \mathbb{Z}$

8. Solve for x: $\tan^2(x) - 1 = 0$
   Solution:
   Substitute $u = \tan(x)$
   $\Rightarrow u^2 - 1 = 0 \quad \Rightarrow u^2 = 1 \quad \Rightarrow u = \pm 1$
   $\Rightarrow \tan(x) = 1$ or $\tan(x) = -1$
   $\arctan(1) = \frac{\pi}{4}$
   $\arctan(-1) = -\frac{\pi}{4}$
   $\Rightarrow x = \frac{\pi}{4} + n\pi$, for $n \in \mathbb{Z}$
   or $x = -\frac{\pi}{4} + n\pi$, for $n \in \mathbb{Z}$

9. Solve for x: $3\tan^2(x) - 1 = 0$
   Solution:
   Substitute $u = \tan(x)$
   $\Rightarrow 3u^2 - 1 = 0 \quad \Rightarrow 3u^2 = 1$
   $\Rightarrow u^2 = \frac{1}{3} \quad \Rightarrow u = \pm\frac{1}{\sqrt{3}} = \pm\frac{\sqrt{3}}{3}$
   $\Rightarrow \tan(x) = \frac{\sqrt{3}}{3}$ or $\tan(x) = -\frac{\sqrt{3}}{3}$
   $\arctan(\frac{\sqrt{3}}{3}) = \frac{\pi}{6}$
   $\arctan(-\frac{\sqrt{3}}{3}) = -\frac{\pi}{6}$
   $\Rightarrow x = \frac{\pi}{6} + n\pi$, for $n \in \mathbb{Z}$
   or $x = -\frac{\pi}{6} + n\pi$, for $n \in \mathbb{Z}$