Trigonometric functions reviewed

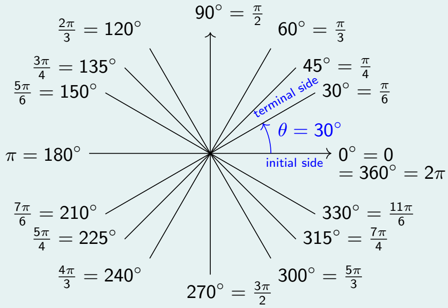
Lesson $\#17$

MAT 1375 Precalculus

New York City College of Technology CUNY

Angles in degree and radian - review

Degree and radian



Conversion between degree and radian

$\boxed{180^\circ = \pi}$ or $\boxed{1^\circ = \frac{\pi}{180}}$

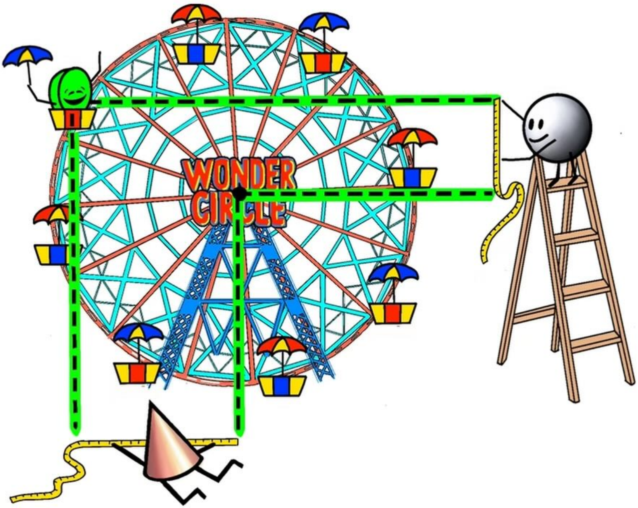
Convert to degree.

1. $\frac{\pi}{3} = \frac{180^\circ}{3} = 60^\circ$
2. $\frac{11\pi}{6} = \frac{11 \cdot 180^\circ}{6} = 11 \cdot 30^\circ = 330^\circ$
3. $\frac{7\pi}{4} = \frac{7 \cdot 180^\circ}{4} = 7 \cdot 45^\circ = 315^\circ$
4. $\frac{5\pi}{4} = \frac{5 \cdot 180^\circ}{4} = \frac{900^\circ}{4} = 225^\circ$
5. $\frac{3\pi}{2} = \frac{3 \cdot 180^\circ}{2} = 3 \cdot 90^\circ = 270^\circ$

Convert to radian.

1. $120^\circ = 120 \cdot \frac{\pi}{180} = \frac{2\pi}{3}$
2. $300^\circ = 300 \cdot \frac{\pi}{180} = \frac{5\pi}{3}$
3. $210^\circ = 210 \cdot \frac{\pi}{180} = \frac{7\pi}{6}$
4. $150^\circ = 150 \cdot \frac{\pi}{180} = \frac{5\pi}{6}$
5. $135^\circ = 135 \cdot \frac{\pi}{180} = \frac{3\pi}{4}$
6. $240^\circ = 240 \cdot \frac{\pi}{180} = \frac{4\pi}{3}$

Trigonometric functions via unit circle - the idea



sin:
The silver sphere figure measures the vertical position of the passenger car from the center.

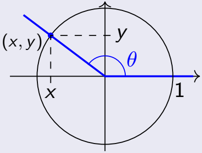
cos:
The copper cone figure measures the horizontal position of the passenger car from the center.

* Note:
The wonder circle has radius 1.

Trigonometric functions via reference angles - review

Definition (Trigonometric functions)

For a point (x, y) on the unit circle, which is on the terminal side of θ:



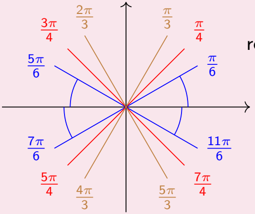
$$\sin(\theta) = y$$

$$\cos(\theta) = x$$

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{y}{x}$$

Reference angle

The **reference angle** is the smallest angle that the terminal side forms with the x-axis.



reference angle:

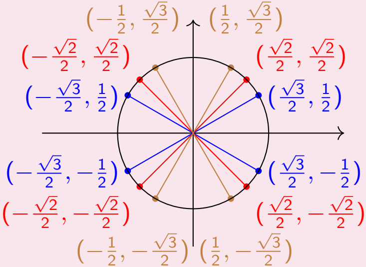
$\frac{\pi}{6}$

$\frac{\pi}{4}$

$\frac{\pi}{3}$

Coordinates on the unit circle

Coordinates coincide with the reference angle up to "±" sign.



Thus: sin, cos, tan of an angle coincide with those of the reference angle except for "±" sign.

Using the calculator

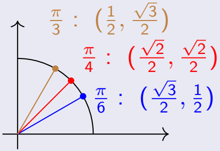
$$\frac{1}{2} = 0.5 \qquad 1 = 1$$

$$\frac{\sqrt{2}}{2} \approx 0.707 \qquad \sqrt{3} \approx 1.732$$

$$\frac{\sqrt{3}}{2} \approx 0.866 \qquad \frac{\sqrt{3}}{3} \approx 0.577$$

Trigonometric functions via reference angles - exercises

Trigonometric functions via reference angle



$\cos\theta = x$

$\sin\theta = y$

$\tan\theta = \frac{y}{x}$

$\frac{1}{2} = 0.5$ $\quad 1 = 1$

$\frac{\sqrt{2}}{2} \approx 0.707$ $\quad \sqrt{3} \approx 1.732$

$\frac{\sqrt{3}}{2} \approx 0.866$ $\quad \frac{\sqrt{3}}{3} \approx 0.577$

1. Find $\sin(\frac{5\pi}{3})$.
   Using calculator:
   $\sin(\frac{5\pi}{3}) \approx -0.866$
   $\Rightarrow \sin(\frac{5\pi}{3}) = -\frac{\sqrt{3}}{2}$
   By hand: Reference angle: $\frac{\pi}{3}$
   $\sin(\frac{5\pi}{3}) = \pm\sin(\frac{\pi}{3}) = \pm\frac{\sqrt{3}}{2}$
   Quadrant of $\frac{5\pi}{3} = 300°$: IV

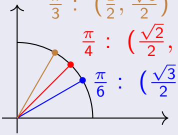
2. Find $\cos(\frac{5\pi}{4})$.
   Calculator:
   $\cos(\frac{5\pi}{4}) \approx -0.707$
   $\Rightarrow \cos(\frac{5\pi}{4}) = -\frac{\sqrt{2}}{2}$
   Reference angle: $\frac{\pi}{4}$
   $\cos(\frac{5\pi}{4}) = \pm\cos(\frac{\pi}{4}) = \pm\frac{\sqrt{2}}{2}$
   Quadrant of $\frac{5\pi}{4} = 225°$: III

3. Find $\sin(-\frac{11\pi}{6})$.
   Calculator:
   $\sin(-\frac{11\pi}{6}) = 0.5$
   $\Rightarrow \sin(-\frac{11\pi}{6}) = \frac{1}{2}$
   Reference angle: $\frac{\pi}{6}$
   $\sin(-\frac{11\pi}{6}) = \pm\sin(\frac{\pi}{6}) = \pm\frac{1}{2}$
   Quadrant of $-\frac{11\pi}{6} = -330°$: I

4. Find $\sin(135°)$.
   Calculator:
   $\sin(135°) \approx 0.707$
   $\Rightarrow \sin(135°) = \frac{\sqrt{2}}{2}$
   $135° = 135 \cdot \frac{\pi}{180} = \frac{3\pi}{4}$
   Reference angle: $\frac{\pi}{4}$
   $\sin(\frac{3\pi}{4}) = \pm\sin(\frac{\pi}{4}) = \pm\frac{\sqrt{2}}{2}$
   Quadrant of $\frac{3\pi}{4} = 135°$: II

5. Find $\cos(210°)$.
   Calculator:
   $\cos(210°) \approx -0.866$
   $\Rightarrow \cos(210°) = -\frac{\sqrt{3}}{2}$
   $210° = 210 \cdot \frac{\pi}{180} = \frac{7\pi}{6}$
   Reference angle: $\frac{\pi}{6}$
   $\cos(\frac{7\pi}{6}) = \pm\cos(\frac{\pi}{6}) = \pm\frac{\sqrt{3}}{2}$
   Quadrant of $\frac{7\pi}{6} = 210°$: III

Trigonometric functions via reference angle



$\frac{\pi}{3}$: $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ $\cos\theta = x$

$\frac{\pi}{4}$: $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ $\sin\theta = y$

$\frac{\pi}{6}$: $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ $\tan\theta = \frac{y}{x}$

$\frac{1}{2} = 0.5$ $1 = 1$

$\frac{\sqrt{2}}{2} \approx 0.707$ $\sqrt{3} \approx 1.732$

$\frac{\sqrt{3}}{2} \approx 0.866$ $\frac{\sqrt{3}}{3} \approx 0.577$

6. Find $\tan(\frac{2\pi}{3})$.

Using calculator:

$\tan(\frac{2\pi}{3}) \approx -1.732$

$\Rightarrow \tan(\frac{2\pi}{3}) = -\sqrt{3}$

Reference angle: $\frac{\pi}{3}$

$\tan(\frac{2\pi}{3}) = \pm\tan(\frac{\pi}{3}) = \pm\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \pm\sqrt{3}$

Quadrant of $\frac{2\pi}{3}$: II

7. Find $\tan(\frac{5\pi}{4})$.

Calculator:

$\tan(\frac{5\pi}{4}) = 1$

$\Rightarrow \tan(\frac{5\pi}{4}) = +1$

Reference angle: $\frac{\pi}{4}$

$\tan(\frac{5\pi}{4}) = \pm\tan(\frac{\pi}{4})$

$= \pm\frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = \pm 1$

Quadrant of $\frac{5\pi}{4}$: III

8. Find $\tan(\frac{5\pi}{6})$.

Calculator:

$\tan(\frac{5\pi}{6}) \approx -0.577$

$\Rightarrow \tan(\frac{5\pi}{6}) = -\frac{\sqrt{3}}{3}$

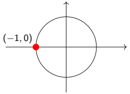
Reference angle: $\frac{\pi}{6}$

$\tan(\frac{5\pi}{6}) = \pm\tan(\frac{\pi}{6}) = \pm\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}$

$= \pm\frac{1}{2}\cdot\frac{2}{\sqrt{3}} = \pm\frac{1}{\sqrt{3}} = \pm\frac{\sqrt{3}}{3}$

Quadrant of $\frac{5\pi}{6}$: II

9. Find:
$\sin(\pi)$, $\cos(\pi)$, $\tan(\pi)$



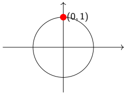
$\Rightarrow \sin(\pi) = 0$

$\Rightarrow \cos(\pi) = -1$

$\Rightarrow \tan(\pi) = \frac{0}{-1} = 0$

(Check with calculator)

10. Find:
$\sin(\frac{\pi}{2})$, $\cos(\frac{\pi}{2})$, $\tan(\frac{\pi}{2})$



$\Rightarrow \sin(\frac{\pi}{2}) = 1$

$\Rightarrow \cos(\frac{\pi}{2}) = 0$

$\Rightarrow \tan(\frac{\pi}{2})$ = "$\frac{1}{0}$" undefined

(Check with calculator)

Sum and difference formulas

Sum and difference formulas

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$$
$$\sin(\alpha-\beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$$
$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$
$$\cos(\alpha-\beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$$
$$\tan(\alpha+\beta) = \frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)}$$
$$\tan(\alpha-\beta) = \frac{\tan(\alpha)-\tan(\beta)}{1+\tan(\alpha)\tan(\beta)}$$

1. Find $\cos(105^\circ)$.

   Note: $105^\circ = 60^\circ + 45^\circ$

   $$\begin{aligned}\cos(105^\circ) &= \cos(60^\circ + 45^\circ)\\ &= \cos(60^\circ)\cos(45^\circ) - \sin(60^\circ)\sin(45^\circ)\\ &= \tfrac{1}{2}\cdot\tfrac{\sqrt{2}}{2} - \tfrac{\sqrt{3}}{2}\cdot\tfrac{\sqrt{2}}{2}\\ &= \tfrac{\sqrt{2}}{4} - \tfrac{\sqrt{3}\cdot\sqrt{2}}{4} = \tfrac{\sqrt{2}-\sqrt{6}}{4}\end{aligned}$$

2. Find $\sin(15^\circ)$.

   Note: $15^\circ = 60^\circ - 45^\circ$

   $$\begin{aligned}\sin(15^\circ) &= \sin(60^\circ - 45^\circ)\\ &= \sin(60^\circ)\cos(45^\circ) - \cos(60^\circ)\sin(45^\circ)\\ &= \tfrac{\sqrt{3}}{2}\cdot\tfrac{\sqrt{2}}{2} - \tfrac{1}{2}\cdot\tfrac{\sqrt{2}}{2}\\ &= \tfrac{\sqrt{3}\cdot\sqrt{2}}{4} - \tfrac{\sqrt{2}}{4}\\ &= \tfrac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

3. Find $\tan(\frac{19\pi}{12})$.

   Note: $\frac{19\pi}{12} = 285^\circ = 60^\circ + 225^\circ$

   $$\begin{aligned}\tan(\tfrac{19\pi}{12}) &= \tan(285^\circ)\\ &= \tan(60^\circ + 225^\circ)\\ &= \tfrac{\tan(60^\circ)+\tan(225^\circ)}{1-\tan(60^\circ)\cdot\tan(225^\circ)}\\ &= \tfrac{\sqrt{3}+1}{1-\sqrt{3}\cdot 1} = \tfrac{1+\sqrt{3}}{1-\sqrt{3}}\end{aligned}$$

Half angle formulas

$$\sin\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1-\cos(\alpha)}{2}}$$

$$\cos\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1+\cos(\alpha)}{2}}$$

$$\tan\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1-\cos(\alpha)}{1+\cos(\alpha)}}$$

Double angle formulas

$$\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha)$$

$$\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha)$$

$$= 1 - 2\sin^2(\alpha) = 2\cos^2(\alpha) - 1$$

$$\tan(2\alpha) = \frac{2\tan(\alpha)}{1-\tan^2(\alpha)}$$

1. Find $\sin(15^\circ)$ using the half angle formulas.

   Note: $15^\circ = \frac{30^\circ}{2}$

   $$\implies \sin(15^\circ) = \sin\left(\frac{30^\circ}{2}\right)$$

   $$= +\sqrt{\frac{1-\cos(30^\circ)}{2}} = \sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}}$$

   $$= \sqrt{\frac{\frac{2-\sqrt{3}}{2}}{2}} = \sqrt{\frac{2-\sqrt{3}}{2}\cdot\frac{1}{2}}$$

   $$= \sqrt{\frac{2-\sqrt{3}}{4}} = \frac{\sqrt{2-\sqrt{3}}}{2}$$

2. Find $\cos\left(\frac{\pi}{8}\right)$ using the half angle formulas.

   Note: $\frac{\pi}{8} = 22.5^\circ = \frac{45^\circ}{2}$

   $$\implies \cos\left(\frac{\pi}{8}\right) = \cos\left(\frac{45^\circ}{2}\right)$$

   $$= +\sqrt{\frac{1+\cos(45^\circ)}{2}} = \sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2}}$$

   $$= \sqrt{\frac{\frac{2+\sqrt{2}}{2}}{2}} = \sqrt{\frac{2+\sqrt{2}}{2}\cdot\frac{1}{2}}$$

   $$= \sqrt{\frac{2+\sqrt{2}}{4}} = \frac{\sqrt{2+\sqrt{2}}}{2}$$