More applications of exp and log

Lesson #16

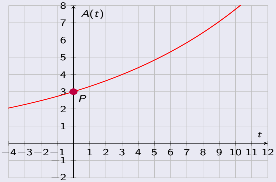
MAT 1375 Precalculus

New York City College of Technology CUNY

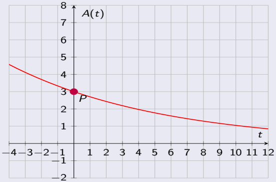
Exponential models in applications

Exponential models

Exponential growth



Exponential decline



Examples:
-Population growth
-Investment at a fixed rate

Examples:
-Population decline
-Radioactive decay

Model: $A(t) = P \cdot e^{r \cdot t}$ or $A(t) = P \cdot (1 + r)^t$

$t =$ time
$P =$ initial amount
$A(t) =$ amount at time t

$r =$ rate of growth or decay
$r > 0$: exponential growth
$r < 0$: exponential decline

Note:
Both
$A(t) = P \cdot e^{r \cdot t}$
and
$A(t) = P \cdot (1 + r)^t$
could be used to model a situation.

For small r, these are close:

$$e^r \approx 1 + r$$

Use the model as stated.
If no particular model is stated use $A = P \cdot e^{r \cdot t}$.

Compound interest

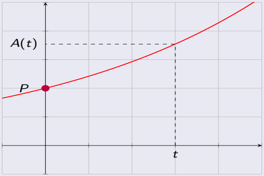
P = principal = (initial investment)

r = (continuous) interest rate

t = time

The current value of the investment, using a *continuous compounding*, is given by:

$$\boxed{A(t) = P \cdot e^{r \cdot t}}$$



1. A \$6,000 investment has an annual interest rate of 3% with a continuous compounding.

(a) How much is the investment worth after 7 years?

(b) How long will it take for the investment to grow to \$10,000?

Solution:

(a) $P = 6,000$ (principal), $r = 3\% = 0.03$

$\Rightarrow A(t) = 6000 \cdot e^{0.03 \cdot t}$

$t = 7 \Rightarrow A(7) = 6000 \cdot e^{0.03 \cdot 7} \approx 7402$

It will be worth \$7,402.

(b) $A(t) = 10,000$

$\Rightarrow 10,000 = 6,000 \cdot e^{0.03 \cdot t}$

$\Rightarrow \frac{10,000}{6,000} = e^{0.03 \cdot t}$

$\Rightarrow \ln(\frac{10,000}{6,000}) = 0.03 \cdot t$

$\Rightarrow t = \frac{\ln(\frac{10,000}{6,000})}{0.03} \approx 17.0$

It will take approximately 17 years.

2. A \$10,000 investment has an annual interest rate of 4.2% with a continuous compounding.

(a) How much is the investment worth after 5 years?

(b) How long will it take for the investment to double?

Solution:

(a) $P = 10,000,\ r = 4.2\% = 0.042$

$\Rightarrow A(t) = 10000 \cdot e^{0.042 \cdot t}$

$t = 5$

$\Rightarrow A(5) = 10000 \cdot e^{0.042 \cdot 5} \approx 12,336.78$

It will be worth \$12,336.78.

(b) $A(t) = 20,000$

$\Rightarrow 20,000 = 10,000 \cdot e^{0.042 \cdot t}$

$\Rightarrow \frac{20,000}{10,000} = e^{0.042 \cdot t} \quad \Rightarrow \ln(2) = 0.042 \cdot t$

$\Rightarrow t = \frac{\ln(2)}{0.042} \approx 16.5$

It will take approximately 16.5 years for the investment to double its value.

Compound n times per year

$$A(t) = P \cdot \left(1 + \frac{r}{n}\right)^{n \cdot t}$$

annual compounding:	$n = 1$
semi-annual compounding:	$n = 2$
quaterly compounding:	$n = 4$
monthly compounding:	$n = 12$

3. \$5,000 is being invested at a rate of 2.7% compounded monthly. The value of the investment after t years is

$$A(t) = 5000 \cdot \left(1 + \frac{0.027}{12}\right)^{12 \cdot t}$$

What will the investment be worth after 15 years?

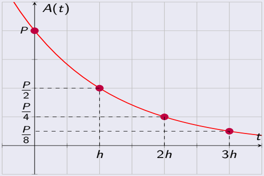
Solution:

$A(15) = 5000 \cdot \left(1 + \frac{0.027}{12}\right)^{12 \cdot 15} \approx 7,493.10$

The investment will be worth \$7,493.10.

Radioactive decay

The *half-life* is the time it takes for a substance to decay to half of its original amount.



$$A(t) = P \cdot e^{r \cdot t}$$

For half-life h:

$$A(h) = \frac{P}{2}$$
$$A(2h) = \frac{P}{4}$$
$$A(3h) = \frac{P}{8}$$
$$\vdots$$

1. A radioactive element has a half-life of 23 minutes. How long will it take for $85g$ of this element to have decayed to $5g$?

Solution:

$P = 85g,\ r = ?$
$\Rightarrow A(t) = 85 \cdot e^{r \cdot t}$

First, find r.
Note: $A(23) = \frac{1}{2} \cdot 85 = 42.5$
$\Rightarrow 42.5 = A(23) = 85 \cdot e^{r \cdot 23}$
$\Rightarrow \frac{42.5}{85} = e^{r \cdot 23}$
$\Rightarrow \ln(\frac{1}{2}) = r \cdot 23$
$\Rightarrow r = \frac{\ln(\frac{1}{2})}{23} \approx -0.03$

Then, for $A(t) = 5$:
$\Rightarrow 5 = 85 \cdot e^{-0.03 \cdot t}$
$\Rightarrow \frac{5}{85} = e^{-0.03 \cdot t}$
$\Rightarrow \ln(\frac{5}{85}) = -0.03 \cdot t$
$\Rightarrow t = \frac{\ln(\frac{5}{85})}{-0.03} \approx 94.4$
It takes about 94.4 minutes.

2. Sodium-24 has a half-life of 14.96 hours. How long does it take for 30*mg* of Sodium-24 to decay to less than 2*mg*?

Solution:

$P = 30g$, $r = ?$
$\Rightarrow A(t) = 30 \cdot e^{r \cdot t}$

First, find r.
Note: $A(14.96) = \frac{1}{2} \cdot 30 = 15$
$\Rightarrow 15 = A(14.96) = 30 \cdot e^{r \cdot 14.96}$
$\Rightarrow 0.5 = e^{r \cdot 14.96}$
$\Rightarrow \ln(0.5) = r \cdot 14.96$
$\Rightarrow r = \frac{\ln(0.5)}{14.96} \approx -0.04633$

Then, for $A(t) = 2$:
$\Rightarrow 2 = 30 \cdot e^{-0.04633 \cdot t}$
$\Rightarrow \frac{2}{30} = e^{-0.04633 \cdot t}$
$\Rightarrow \ln(\frac{2}{30}) = -0.04633 \cdot t$
$\Rightarrow t = \frac{\ln(\frac{2}{30})}{-0.04633} \approx 58.45$
It takes about 58.45 hours.

3. How much is left after 26 weeks of 4.7*lb* of a radioactive substance, if the half-life of the substance is 8.9 weeks?

Solution:

$P = 4.7$, $r = ?$
$\Rightarrow A(t) = 4.7 \cdot e^{r \cdot t}$

First, find r.
Note: $A(8.9) = \frac{1}{2} \cdot 4.7 = 2.35$
$\Rightarrow 2.35 = A(8.9) = 4.7 \cdot e^{r \cdot 8.9}$
$\Rightarrow 0.5 = e^{r \cdot 8.9}$
$\Rightarrow \ln(0.5) = r \cdot 8.9$
$\Rightarrow r = \frac{\ln(0.5)}{8.9} \approx -0.07788$
$\Rightarrow A(t) = 4.7 \cdot e^{-0.07788 \cdot t}$

Then, for $t = 26$:
$\Rightarrow A(26) = 4.7 \cdot e^{-0.07788 \cdot 26} \approx 0.62$
After 26 weeks, only 0.62*lb* of the substance will be left.