# More applications of exp and log

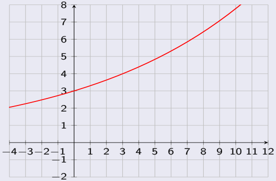
Lesson #16

MAT 1375 Precalculus

New York City College of Technology CUNY

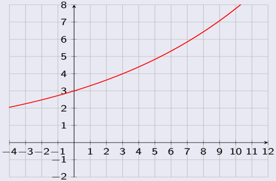
## Exponential models

Exponential growth



## Exponential models

Exponential growth




**Examples:**

-Population growth

## Exponential models

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**Examples:**

-Population growth

-Investment at a fixed rate

## Exponential models

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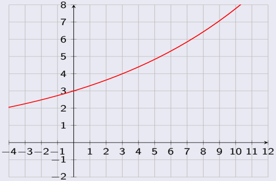


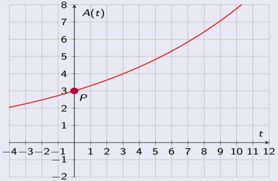

**Examples:**
-Population growth
-Investment at a fixed rate

**Model:** $A(t) = P \cdot e^{r \cdot t}$ or $A(t) = P \cdot (1 + r)^t$

# Exponential models in applications

## Exponential models

Exponential growth

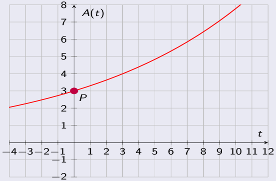



**Examples:**
-Population growth
-Investment at a fixed rate

**Model:** $A(t) = P \cdot e^{r \cdot t}$ or $A(t) = P \cdot (1 + r)^t$
$t =$ time
$P =$ initial amount
$A(t) =$ amount at time $t$

## Exponential models

Exponential growth




**Examples:**
-Population growth
-Investment at a fixed rate

**Model:** $A(t) = P \cdot e^{r \cdot t}$ or $A(t) = P \cdot (1 + r)^t$

$t =$ time

$r =$ rate of growth

$P =$ initial amount

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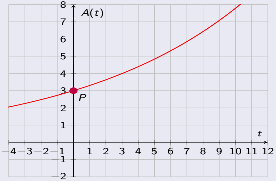
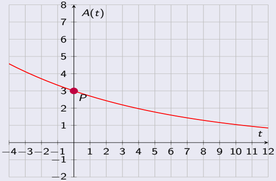
## Exponential models

Exponential growth




Exponential decline



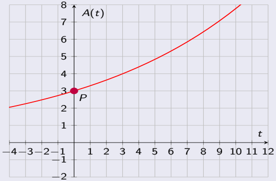
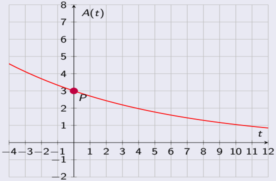

**Examples:**
-Population growth
-Investment at a fixed rate

**Model:** $A(t) = P \cdot e^{r \cdot t}$ or $A(t) = P \cdot (1 + r)^t$

$t =$ time
$P =$ initial amount
$A(t) =$ amount at time $t$

$r =$ rate of growth or decay
$r > 0$ : exponential growth
$r < 0$ : exponential decline

## Exponential models

| Exponential growth | Exponential decline |
|---|---|
|  |  |





**Examples:**
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-Investment at a fixed rate

**Examples:**
-Population decline

**Model:** $A(t) = P \cdot e^{r \cdot t}$ or $A(t) = P \cdot (1 + r)^t$

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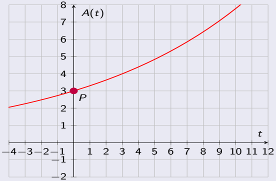
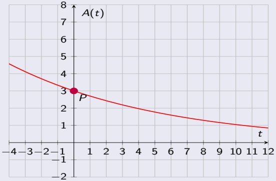
# Exponential models in applications

## Exponential models

**Exponential growth**




**Exponential decline**




**Examples:**
-Population growth
-Investment at a fixed rate

**Examples:**
-Population decline
-Radioactive decay

**Model:** $A(t) = P \cdot e^{r \cdot t}$ or $A(t) = P \cdot (1+r)^t$

$t =$ time
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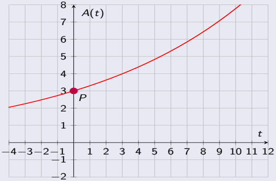
# Exponential models in applications

## Exponential models

Exponential growth

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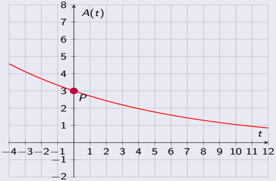



**Examples:**
-Population growth
-Investment at a fixed rate

**Examples:**
-Population decline
-Radioactive decay

**Model:** $A(t) = P \cdot e^{r \cdot t}$ or $A(t) = P \cdot (1 + r)^t$

$t =$ time
$P =$ initial amount
$A(t) =$ amount at time $t$

$r =$ rate of growth or decay
$r > 0$ : exponential growth
$r < 0$ : exponential decline

**Note:**
Both
$A(t) = P \cdot e^{r \cdot t}$
and
$A(t) = P \cdot (1 + r)^t$
could be used to model a situation.

For small $r$, these are close:

$$e^r \approx 1 + r$$

Use the model as stated.
If no particular model is stated use $A = P \cdot e^{r \cdot t}$.

## Compound interest

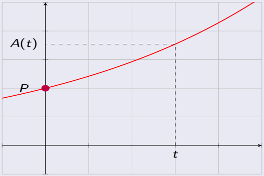
$P$ = principal = (initial investment)

$r$ = (continuous) interest rate

$t$ = time

The current value of the investment, using a *continuous compounding*, is given by:

$$\boxed{A(t) = P \cdot e^{r \cdot t}}$$



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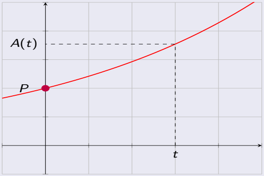


1. A $6,000 investment has an annual interest rate of 3% with a continuous compounding.

(a) How much is the investment worth after 7 years?

(b) How long will it take for the investment to grow to $10,000?

# Compound interest on an investment

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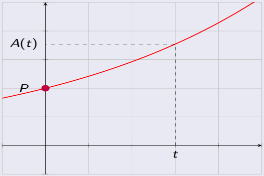


1. A \$6,000 investment has an annual interest rate of 3% with a continuous compounding.

(a) How much is the investment worth after 7 years?

(b) How long will it take for the investment to grow to \$10,000?

**Solution:**

(a) $P = 6,000$ (principal), $r = 3\% = 0.03$
$\Rightarrow A(t) = 6000 \cdot e^{0.03 \cdot t}$
$t = 7 \Rightarrow A(7) = 6000 \cdot e^{0.03 \cdot 7} \approx 7402$
It will be worth \$7,402.

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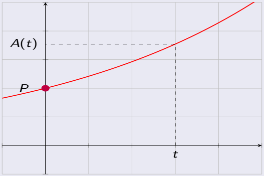


1. A \$6,000 investment has an annual interest rate of 3% with a continuous compounding.

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It will be worth \$7,402.

(b) $A(t) = 10,000$

$\Rightarrow 10,000 = 6,000 \cdot e^{0.03 \cdot t}$

$\Rightarrow \frac{10{,}000}{6{,}000} = e^{0.03 \cdot t}$

$\Rightarrow \ln(\frac{10{,}000}{6{,}000}) = 0.03 \cdot t$

$\Rightarrow t = \frac{\ln(\frac{10{,}000}{6{,}000})}{0.03} \approx 17.0$

It will take approximately 17 years.

2. A $\$10,000$ investment has an annual interest rate of $4.2\%$ with a continuous compounding.

(a) How much is the investment worth after 5 years?

(b) How long will it take for the investment to double?

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(b) How long will it take for the investment to double?

**Solution:**

(a) $P = 10,000,\ r = 4.2\% = 0.042$

$\Rightarrow A(t) = 10000 \cdot e^{0.042 \cdot t}$

$t = 5$

$\Rightarrow A(5) = 10000 \cdot e^{0.042 \cdot 5} \approx 12,336.78$

It will be worth $\$12,336.78$.

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It will be worth $\$12,336.78$.

(b) $A(t) = 20,000$

$\Rightarrow 20,000 = 10,000 \cdot e^{0.042 \cdot t}$

$\Rightarrow \frac{20,000}{10,000} = e^{0.042 \cdot t} \quad \Rightarrow \ln(2) = 0.042 \cdot t$

$\Rightarrow t = \frac{\ln(2)}{0.042} \approx 16.5$

It will take approximately 16.5 years for the investment to double its value.

# Compound interest - exercises

2. A $\$10,000$ investment has an annual interest rate of 4.2% with a continuous compounding.

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It will take approximately 16.5 years for the investment to double its value.

**Compound $n$ times per year**

$$A(t) = P \cdot \left(1 + \frac{r}{n}\right)^{n \cdot t}$$

| | |
|---|---|
| *annual* compounding: | $n = 1$ |
| *semi-annual* compounding: | $n = 2$ |
| *quaterly* compounding: | $n = 4$ |
| *monthly* compounding: | $n = 12$ |

2. A $\$10,000$ investment has an annual interest rate of 4.2% with a continuous compounding.

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*annual* compounding: $n = 1$
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*monthly* compounding: $n = 12$

3. $\$5,000$ is being invested at a rate of 2.7% compounded monthly. The value of the investment after $t$ years is

$$A(t) = 5000 \cdot \left(1 + \frac{0.027}{12}\right)^{12 \cdot t}$$

What will the investment be worth after 15 years?

2. A $\$10,000$ investment has an annual interest rate of 4.2% with a continuous compounding.

(a) How much is the investment worth after 5 years?

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### Compound $n$ times per year

$$A(t) = P \cdot \left(1 + \frac{r}{n}\right)^{n \cdot t}$$

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$$A(t) = 5000 \cdot \left(1 + \frac{0.027}{12}\right)^{12 \cdot t}$$

What will the investment be worth after 15 years?

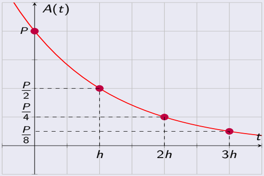
**Solution:**

$A(15) = 5000 \cdot \left(1 + \frac{0.027}{12}\right)^{12 \cdot 15} \approx 7,493.10$

The investment will be worth $\$7,493.10$.

## Radioactive decay

The *half-life* is the time it takes for a substance to decay to half of its original amount.




$$\boxed{A(t) = P \cdot e^{r \cdot t}}$$

For half-life $h$:

$$A(h) = \frac{P}{2}$$
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$$\vdots$$

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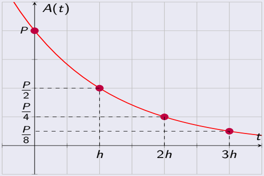


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1. A radioactive element has a half-life of 23 minutes. How long will it take for $85g$ of this element to have decayed to $5g$?

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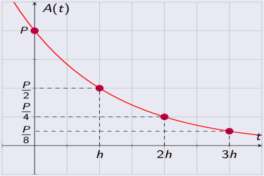


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$$\vdots$$

1. A radioactive element has a half-life of 23 minutes. How long will it take for $85g$ of this element to have decayed to $5g$?

**Solution:**

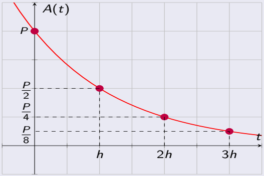
$P = 85g, \; r = ?$
$\Rightarrow A(t) = 85 \cdot e^{r \cdot t}$

First, find $r$.
Note: $A(23) = \frac{1}{2} \cdot 85 = 42.5$

$\Rightarrow 42.5 = A(23) = 85 \cdot e^{r \cdot 23}$
$\Rightarrow \frac{42.5}{85} = e^{r \cdot 23}$
$\Rightarrow \ln(\frac{1}{2}) = r \cdot 23$
$\Rightarrow r = \frac{\ln(\frac{1}{2})}{23} \approx -0.03$

## Radioactive decay

The *half-life* is the time it takes for a substance to decay to half of its original amount.

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$$A(h) = \frac{P}{2}$$
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1. A radioactive element has a half-life of 23 minutes. How long will it take for $85g$ of this element to have decayed to $5g$?

**Solution:**

$P = 85g$, $r = ?$
$\Rightarrow A(t) = 85 \cdot e^{r \cdot t}$

First, find $r$.
Note: $A(23) = \frac{1}{2} \cdot 85 = 42.5$
$\Rightarrow 42.5 = A(23) = 85 \cdot e^{r \cdot 23}$
$\Rightarrow \frac{42.5}{85} = e^{r \cdot 23}$
$\Rightarrow \ln(\frac{1}{2}) = r \cdot 23$
$\Rightarrow r = \frac{\ln(\frac{1}{2})}{23} \approx -0.03$

Then, for $A(t) = 5$:
$\Rightarrow 5 = 85 \cdot e^{-0.03 \cdot t}$
$\Rightarrow \frac{5}{85} = e^{-0.03 \cdot t}$
$\Rightarrow \ln(\frac{5}{85}) = -0.03 \cdot t$
$\Rightarrow t = \frac{\ln(\frac{5}{85})}{-0.03} \approx 94.4$
It takes about 94.4 minutes.

2. Sodium-24 has a half-life of 14.96 hours. How long does it take for $30mg$ of Sodium-24 to decay to less than $2mg$?

3. How much is left after 26 weeks of $4.7lb$ of a radioactive substance, if the half-life of the substance is 8.9 weeks?

# Radioactive decay - exercises

2. Sodium-24 has a half-life of 14.96 hours. How long does it take for 30$mg$ of Sodium-24 to decay to less than 2$mg$?

**Solution:**

$P = 30g$, $r = ?$
$\Rightarrow A(t) = 30 \cdot e^{r \cdot t}$

First, find $r$.
Note: $A(14.96) = \frac{1}{2} \cdot 30 = 15$
$\Rightarrow 15 = A(14.96) = 30 \cdot e^{r \cdot 14.96}$
$\Rightarrow 0.5 = e^{r \cdot 14.96}$
$\Rightarrow \ln(0.5) = r \cdot 14.96$
$\Rightarrow r = \frac{\ln(0.5)}{14.96} \approx -0.04633$

Then, for $A(t) = 2$:
$\Rightarrow 2 = 30 \cdot e^{-0.04633 \cdot t}$
$\Rightarrow \frac{2}{30} = e^{-0.04633 \cdot t}$
$\Rightarrow \ln(\frac{2}{30}) = -0.04633 \cdot t$
$\Rightarrow t = \frac{\ln(\frac{2}{30})}{-0.04633} \approx 58.45$
It takes about 58.45 hours.

3. How much is left after 26 weeks of 4.7$lb$ of a radioactive substance, if the half-life of the substance is 8.9 weeks?

2. Sodium-24 has a half-life of 14.96 hours. How long does it take for 30$mg$ of Sodium-24 to decay to less than 2$mg$?

**Solution:**

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$\Rightarrow A(t) = 30 \cdot e^{r \cdot t}$

First, find $r$.
Note: $A(14.96) = \frac{1}{2} \cdot 30 = 15$
$\Rightarrow 15 = A(14.96) = 30 \cdot e^{r \cdot 14.96}$
$\Rightarrow 0.5 = e^{r \cdot 14.96}$
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Then, for $A(t) = 2$:
$\Rightarrow 2 = 30 \cdot e^{-0.04633 \cdot t}$
$\Rightarrow \frac{2}{30} = e^{-0.04633 \cdot t}$
$\Rightarrow \ln(\frac{2}{30}) = -0.04633 \cdot t$
$\Rightarrow t = \frac{\ln(\frac{2}{30})}{-0.04633} \approx 58.45$
It takes about 58.45 hours.

3. How much is left after 26 weeks of 4.7$lb$ of a radioactive substance, if the half-life of the substance is 8.9 weeks?

**Solution:**

$P = 4.7$, $r = ?$
$\Rightarrow A(t) = 4.7 \cdot e^{r \cdot t}$

First, find $r$.
Note: $A(8.9) = \frac{1}{2} \cdot 4.7 = 2.35$
$\Rightarrow 2.35 = A(8.9) = 4.7 \cdot e^{r \cdot 8.9}$
$\Rightarrow 0.5 = e^{r \cdot 8.9}$
$\Rightarrow \ln(0.5) = r \cdot 8.9$
$\Rightarrow r = \frac{\ln(0.5)}{8.9} \approx -0.07788$
$\Rightarrow A(t) = 4.7 \cdot e^{-0.07788 \cdot t}$

Then, for $t = 26$:
$\Rightarrow A(26) = 4.7 \cdot e^{-0.07788 \cdot 26} \approx 0.62$
After 26 weeks, only 0.62$lb$ of the substance will be left.