Exponential equations and applications

Lesson $\#15$

MAT 1375 Precalculus

New York City College of Technology CUNY

Solving exponential equations - exercises

Solve the exponential equation.

1.
$$2^{x+3} = 16$$
$$\Rightarrow 2^{x+3} = 2^4$$
$$\Rightarrow x + 3 = 4$$
$$\Rightarrow x = 1$$

Exponential function is one-to-one

$$B^r = B^s \quad \Leftrightarrow \quad r = s$$

2.
$$5^{2x-6} = 25$$
$$\Rightarrow 5^{2x-6} = 5^2$$
$$\Rightarrow 2x - 6 = 2$$
$$\Rightarrow 2x = 8$$
$$\Rightarrow x = 4$$

3.
$$6^{x+7} = 36^x$$
$$\Rightarrow 6^{x+7} = 6^{2x}$$
$$\Rightarrow x + 7 = 2x$$
$$\Rightarrow 7 = x$$

4.
$$8^{x-7} = 4^{3x+5}$$
$$\Rightarrow 2^{3(x-7)} = 2^{2(3x+5)}$$
$$\Rightarrow 3(x-7) = 2(3x+5)$$
$$\Rightarrow 3x - 21 = 6x + 10$$
$$\Rightarrow -3x = 31$$
$$\Rightarrow x = -\frac{31}{3}$$

5.
$$2^x = 7$$
$$\Rightarrow \log(2^x) = \log(7)$$
$$\Rightarrow x \cdot \log(2) = \log(7)$$
$$\Rightarrow x = \frac{\log(7)}{\log(2)}$$

Solving exponential equations - exercises

Solve the exponential equation.

6.
$$3^{x+5} = 8$$
$$\Rightarrow \log(3^{x+5}) = \log(8)$$
$$\Rightarrow (x+5)\cdot\log 3 = \log 8$$
$$\Rightarrow x\cdot\log 3 + 5\cdot\log 3 = \log 8$$
$$\Rightarrow x\cdot\log 3 = \log 8 - 5\cdot\log 3$$
$$\Rightarrow x = \frac{\log 8 - 5\cdot\log 3}{\log 3}$$

7.
$$5^{2x+6} = 2^{x+9}$$
$$\Rightarrow \log(5^{2x+6}) = \log(2^{x+9})$$
$$\Rightarrow (2x+6)\cdot\log 5 = (x+9)\cdot\log 2$$
$$\Rightarrow 2x\cdot\log 5 + 6\cdot\log 5 = x\cdot\log 2 + 9\cdot\log 2$$
$$\Rightarrow 2x\cdot\log 5 - x\cdot\log 2 = 9\cdot\log 2 - 6\cdot\log 5$$
$$\Rightarrow x\cdot(2\log 5 - \log 2) = 9\cdot\log 2 - 6\cdot\log 5$$
$$\Rightarrow x = \frac{9\cdot\log 2 - 6\cdot\log 5}{2\log 5 - \log 2}$$

8.
$$3^{2x+4} = 6^{5-x}$$
$$\Rightarrow (2x+4)\log 3 = (5-x)\log 6$$
$$\Rightarrow 2x\log 3 + 4\log 3 = 5\log 6 - x\log 6$$
$$\Rightarrow 2x\log 3 + x\log 6 = 5\log 6 - 4\log 3$$
$$\Rightarrow x\cdot(2\log 3 + \log 6) = 5\log 6 - 4\log 3$$
$$\Rightarrow x = \frac{5\log 6 - 4\log 3}{2\log 3 + \log 6}$$

9.
$$2^{3-4x} = 7^{6x-5}$$
$$\Rightarrow (3-4x)\log 2 = (6x-5)\log 7$$
$$\Rightarrow 3\log 2 - 4x\log 2 = 6x\log 7 - 5\log 7$$
$$\Rightarrow 3\log 2 + 5\log 7 = 6x\log 7 + 4x\log 2$$
$$\Rightarrow 3\log 2 + 5\log 7 = x\cdot(6\log 7 + 4\log 2)$$
$$\Rightarrow x = \frac{3\log 2 + 5\log 7}{6\log 7 + 4\log 2}$$

Solve the exponential equation.

10. $2e^x = 5$

$\Rightarrow e^x = \frac{5}{2}$

$\Rightarrow \ln(e^x) = \ln(\frac{5}{2})$

$\Rightarrow x \cdot \ln(e) = \ln(\frac{5}{2})$

Recall: $\boxed{\ln(e) = 1}$

$\Rightarrow x = \ln(\frac{5}{2})$

11. $4e^{5x-12} = 9$

$\Rightarrow e^{5x-12} = \frac{9}{4}$

$\Rightarrow \ln(e^{5x-12}) = \ln(\frac{9}{4})$

$\Rightarrow (5x-12) \cdot \ln(e) = \ln(\frac{9}{4})$

$\Rightarrow 5x - 12 = \ln(\frac{9}{4})$

$\Rightarrow 5x = \ln(\frac{9}{4}) + 12$

$\Rightarrow x = \frac{\ln(\frac{9}{4})+12}{5}$

Solve the exponential equation.

12. $\frac{4e^x+6}{7e^x-3} = 12$

Multiply both sides by the denominator $7e^x - 3$:

$\Rightarrow 4e^x + 6 = 12 \cdot (7e^x - 3)$

$\Rightarrow 4e^x + 6 = 84e^x - 36$

Separate e^x to one side:

$\Rightarrow 4e^x - 84e^x = -36 - 6$

$\Rightarrow -80e^x = -42$

$\Rightarrow e^x = \frac{-42}{-80} = \frac{21}{40}$

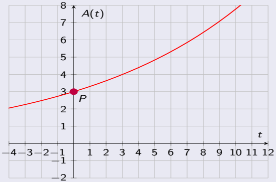
Solve for x:

$\Rightarrow \ln(e^x) = \ln(\frac{21}{40})$

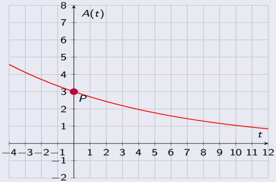
$\Rightarrow x = \ln(\frac{21}{40})$

Exponential models

Exponential growth



Exponential decline



Examples:
-Population growth
-Investment at a fixed rate

Examples:
-Population decline
-Radioactive decay

Model: $A(t) = P \cdot e^{r \cdot t}$ or $A(t) = P \cdot (1 + r)^t$

$t =$ time
$P =$ initial amount
$A(t) =$ amount at time t

$r =$ rate of growth or decay
$r > 0$: exponential growth
$r < 0$: exponential decline

Note:
Both
$A(t) = P \cdot e^{r \cdot t}$
and
$A(t) = P \cdot (1 + r)^t$
could be used to model a situation.

For small r, these are close:

$$e^r \approx 1 + r$$

Use the model as stated.
If no particular model is stated use $A = P \cdot e^{r \cdot t}$.

Exponential models - exercises

1. A city has a population size of $56,000$ people. The city grows exponentially at a rate of 6% per year.

(a) What is the population size after 15 years?

(b) How long will it take for the population to reach $200,000$?

Solution:

(a) $P = 56,000,\ r = 6\% = 0.06$

$\Rightarrow A(t) = 56000 \cdot e^{0.06 \cdot t}$

$t = 15$

$\Rightarrow A(15) = 56000 \cdot e^{0.06 \cdot 15} \approx 138,000$

There will be approximately $138,000$ people in the city.

(b) $A(t) = 200,000$

$\Rightarrow 200,000 = 56,000 \cdot e^{0.06 \cdot t}$

$\Rightarrow \frac{200{,}000}{56{,}000} = e^{0.06 \cdot t}$

$\Rightarrow \ln(\frac{200{,}000}{56{,}000}) = 0.06 \cdot t$

$\Rightarrow t = \frac{\ln(\frac{200{,}000}{56{,}000})}{0.06} \approx 21.2$

It will take approximately 21.2 years.

2. The world population in 2020 consists of about 7.8 billion people and grows at a rate of 1.1% per year.

(a) Assuming constant growth, what will the world population be in 2075?

(b) How long does it take for the world population to triple?

Solution:

(a) $P = 7.8[\text{billion}],\ r = 1.1\% = 0.011$

$\Rightarrow A(t) = 7.8 \cdot e^{0.011 \cdot t}$

$t = 2075 - 2020 = 55$

$\Rightarrow A(55) = 7.8 \cdot e^{0.011 \cdot 55} \approx 14.3$

There will be approximately 14.3 billion people.

(b) $A(t) = 3 \cdot 7.8$

$\Rightarrow 3 \cdot 7.8 = 7.8 \cdot e^{0.011 \cdot t}$

$\Rightarrow 3 = e^{0.011 \cdot t}$

$\Rightarrow \ln(3) = 0.011 \cdot t$

$\Rightarrow t = \frac{\ln(3)}{0.011} \approx 99.87$

It takes about 99.87 years for the world population to triple in size.

Exponential models - exercises

3. An ant colony of 4900 ants decreases exponentially at a rate of 3.5% per month.

(a) How many ants will be left after 2 years?

(b) When will only half of the ants be left?

Solution:

(a) $P = 4900$, $r = -3.5\% = -0.035$

$\Rightarrow A(t) = 4900 \cdot e^{-0.035 \cdot t}$

$t = 2 \cdot 12 = 24$ [months]

$\Rightarrow A(24) = 4900 \cdot e^{-0.035 \cdot 24} \approx 2100$

After 2 years, there will be approximately 2100 ants left.

(b) $A(t) = \frac{1}{2} \cdot 4900 = 2450$

$\Rightarrow 2450 = 4900 \cdot e^{-0.035 \cdot t}$

$\Rightarrow \frac{1}{2} = e^{-0.035 \cdot t}$

$\Rightarrow \ln(\frac{1}{2}) = -0.035 \cdot t$

$\Rightarrow t = \frac{\ln(\frac{1}{2})}{-0.035} \approx 19.8$

It will take approximately 19.8 months until half of the ants will be left.

4. The number of bacteria in a bacteria culture is modeled with by the exponential function $A(t) = P \cdot (1 + r)^t$. If the bacteria culture grows exponentially at a rate of 12.7% per day, then how long do you have to wait for the culture to double?

Solution:

$P =?$ (unknown)

$r = 12.7\% = 0.127$

$\Rightarrow A(t) = P \cdot (1 + 0.127)^t = P \cdot 1.127^t$

$A(t) = 2 \cdot P$

$\Rightarrow 2 \cdot P = P \cdot 1.127^t$

$\Rightarrow 2 = 1.127^t$

$\Rightarrow \ln(2) = t \cdot \ln(1.127)$

$\Rightarrow t = \frac{\ln(2)}{\ln(1.127)} \approx 5.8$

It takes approximately 5.8 days for the bacteria colony to double in size.

Note: It takes the same amount of time for the culture to double no matter what the initial number P of bacterias was.