Properties of log and logarithmic equations

Lesson #14

MAT 1375 Precalculus

New York City College of Technology CUNY

Algebraic properties of exp and log - review

Recall the following exponential identities:

$$B^r \cdot B^s = B^{r+s}$$
$$\frac{B^r}{B^s} = B^{r-s}$$
$$(B^r)^s = B^{r \cdot s}$$

These translate into logarithmic identities:

Logarithmic idenitities

$$\log_B(r \cdot s) = \log_B(r) + \log_B(s)$$
$$\log_B\left(\frac{r}{s}\right) = \log_B(r) - \log_B(s)$$
$$\log_B(r^s) = s \cdot \log_B(r)$$

Combine to one logarithm:

1. $$\begin{aligned} &2\log(x) + 3\log(y) - 5\log(z) \\ &= \log(x^2) + \log(y^3) - \log(z^5) \\ &= \log(x^2 \cdot y^3) - \log(z^5) \\ &= \log\left(\frac{x^2 \cdot y^3}{z^5}\right) \end{aligned}$$
2. $$\begin{aligned} &4\log_5(x) + 2\log_5(y) + 6\log_5(z) \\ &= \log_5(x^4) + \log_5(y^2) + \log_5(z^6) \\ &= \log_5(x^4 \cdot y^2 \cdot z^6) \end{aligned}$$
3. $$\begin{aligned} &5\log(x) - 7\log(y) - 4\log(z) \\ &= \log(x^5) - \log(y^7) - \log(z^4) \\ &= \log(\tfrac{x^5}{y^7}) - \log(z^4) \\ &= \log\left(\frac{x^5}{y^7 \cdot z^4}\right) \end{aligned}$$

Combining and expanding logarithms - exercises

4. $\frac{1}{2}\ln(x) - \frac{3}{5}\ln(y)$
$= \ln(x^{\frac{1}{2}}) - \ln(y^{\frac{3}{5}})$
$= \ln(\sqrt{x}) - \ln(\sqrt[5]{y^3})$
$= \ln\left(\frac{\sqrt{x}}{\sqrt[5]{y^3}}\right)$

Recall:
$x^{\frac{1}{n}} = \sqrt[n]{x}$ and $x^{\frac{m}{n}} = \sqrt[n]{x^m}$

Expand as much as possible:

5. $\log_7(x^3 \cdot y^8)$
$= \log_7(x^3) + \log_7(y^8)$
$= 3\log_7(x) + 8\log_7(y)$

6. $\log\left(\frac{3x}{y}\right)$
$= \log(3x) - \log(y)$
$= \log(3) + \log(x) - \log(y)$

7. $\log\left(\frac{5x}{y^4}\right)$
$= \log(5x) - \log(y^4)$
$= \log(5) + \log(x) - 4\log(y)$

8. $\log_5\left(\frac{x^2}{y^3 \cdot z^4}\right)$
$= \log_5(x^2) - \log_5(y^3 \cdot z^4)$
$= 2\log_5(x) - (3\log_5(y) + 4\log_5(z))$
$= 2\log_5(x) - 3\log_5(y) - 4\log_5(z)$

9. $\log_2\left(\frac{\sqrt{x} \cdot y^7}{\sqrt[3]{z}}\right) = \log_2\left(\frac{x^{\frac{1}{2}} \cdot y^7}{z^{\frac{1}{3}}}\right)$
$= \log_2(x^{\frac{1}{2}}) + \log_2(y^7) - \log_2(z^{\frac{1}{3}})$
$= \frac{1}{2} \cdot \log_2(x) + 7 \cdot \log_2(y) - \frac{1}{3} \cdot \log_2(z)$

10. $\log\left(\sqrt{\frac{x^9}{y^7 \cdot z^5}}\right) = \log\left(\frac{x^9}{y^7 \cdot z^5}\right)^{\frac{1}{2}} = \frac{1}{2} \cdot \log\left(\frac{x^9}{y^7 \cdot z^5}\right)$
$= \frac{1}{2} \cdot \Big(9\log(x) - 7\log(y) - 5 \cdot \log(z)\Big)$
$= \frac{9}{2}\log(x) - \frac{7}{2}\log(y) - \frac{5}{2}\log(z)$

Solving logarithmic equations - exercises

Solve the logarithmic equation.

1. $\log_4(2x+3) = \log_4(7x-5)$
   $\Rightarrow 2x+3 = 7x-5$
   $\Rightarrow -5x = -8$
   $\Rightarrow x = \frac{-8}{-5} = \frac{8}{5}$

The logarithm is one-to-one

$$\log_B(r) = \log_B(s) \quad \Leftrightarrow \quad r = s$$

2. $\log_7(2x+4) = \log_7(6x-3)$
   $\Rightarrow 2x+4 = 6x-3$
   $\Rightarrow -4x = -7$
   $\Rightarrow x = \frac{7}{4}$

3. $\ln(2x-7) = \ln(x-4)$
   $\Rightarrow 2x-7 = x-4 \qquad \Rightarrow x = 3$ ✗
   Note that $x = 3$ is **NOT** a solution, because $\ln(2\cdot 3-7) = \ln(-1)$ ✗ is undefined!

More precisely:

The logarithm is one-to-one

When $r > 0$ and $s > 0$:

$$\log_B(r) = \log_B(s) \quad \Leftrightarrow \quad r = s$$

4. $\log_3(3x-8) = \log_3(2x-6)$
   $\Rightarrow 3x-8 = 2x-6$
   $\Rightarrow x = 2$ ✗ not a solution:
   $\log_3(3\cdot 2-8) = \log_3(-2)$ is undefined
   $\log_3(2\cdot 2-6) = \log_3(-2)$ is undefined
   $\Rightarrow$ There is no solution!

5. $\log(2x+11) = \log(4x+17)$
   $\Rightarrow 2x+11 = 4x+17$
   $\Rightarrow -6 = 2x$
   $\Rightarrow x = -3$ ✓ is a solution:
   $\log(2\cdot(-3)+11) = \log(5)$ is defined
   $\log(4\cdot(-3)+17) = \log(5)$ is defined

Solving logarithmic equations - exercises

Solve the logarithmic equation.

1. $\log_3(x) = 2$
   $\Rightarrow x = 3^2 = 9$ ✓

Recall

$$\log_B(x) = y \quad \Leftrightarrow \quad x = B^y$$

2. $\log_5(x) = 7$
   $\Rightarrow x = 5^7 = 78125$ ✓

3. $\log_2(x+5) = 4$
   $\Rightarrow x + 5 = 2^4 = 16$
   $\Rightarrow x = 16 - 5 = 11$ ✓

4. $\log_7(5x+8) = 3$
   $\Rightarrow 5x + 8 = 7^3 = 343$
   $\Rightarrow 5x = 335$
   $\Rightarrow x = 67$ ✓

5. $\log_2(x+1) + \log_2(x-3) = 5$
   $\Rightarrow \log_2((x+1)\cdot(x-3)) = 5$
   $\Rightarrow (x+1)\cdot(x-3) = 2^5$
   $\Rightarrow x^2 - 2x - 3 = 32$
   $\Rightarrow x^2 - 2x - 35 = 0$
   $\Rightarrow (x+5)(x-7) = 0$
   $\Rightarrow x = -5$ ✗ or $x = 7$ ✓
   $\Rightarrow x = 7$

6. $\log_3(x-2) + \log_3(x+6) = 2$
   $\Rightarrow (x-2)(x+6) = 3^2$
   $\Rightarrow x^2 + 4x - 12 = 9$
   $\Rightarrow x^2 + 4x - 21 = 0$
   $\Rightarrow (x+7)(x-3) = 0$
   $\Rightarrow x = -7$ ✗ or $x = 3$ ✓
   $\Rightarrow x = 3$

Solving logarithmic equations - exercises

Solve the logarithmic equation.

7.

$\log_5(x+3) + \log_5(x+4) = \log_5(2)$
$\Rightarrow \log_5((x+3)\cdot(x+4)) = \log_5(2)$
$\Rightarrow (x+3)\cdot(x+4) = 2$
$\Rightarrow x^2 + 7x + 12 = 2$
$\Rightarrow x^2 + 7x + 10 = 0$
$\Rightarrow (x+2)(x+5) = 0$
$\Rightarrow x = -2$ ✓ or $x = -5$ ✗
$\Rightarrow x = -2$

8.

$\log_3(x-7) + \log_3(2-x) = \log_3(4)$
$\Rightarrow \log_3((x-7)\cdot(2-x)) = \log_3(4)$
$\Rightarrow (x-7)\cdot(2-x) = 4$
$\Rightarrow 2x - x^2 - 14 + 7x = 4$
$\Rightarrow -x^2 + 9x - 18 = 0$
$\overset{\cdot(-1)}{\Rightarrow} x^2 - 9x + 18 = 0$
$\Rightarrow (x-3)\cdot(x-6) = 0$
$\Rightarrow x = 3$ ✗ or $x = 6$ ✗
$\Rightarrow$ There is no solution!

9.

$\ln(x+1) + \ln(7-x) = \ln(12)$
$\Rightarrow \ln((x+1)\cdot(7-x)) = \ln(21)$
$\Rightarrow (x+1)\cdot(7-x) = 12$
$\Rightarrow 7x - x^2 + 7 - x = 12$
$\Rightarrow -x^2 + 6x - 5 = 0$
$\Rightarrow x^2 - 6x + 5 = 0$
$\Rightarrow (x-1)\cdot(x-5) = 0$
$\Rightarrow x = 1$ ✓ or $x = 5$ ✓
$\Rightarrow$ There are two solutions!

10.

$\log_4(x+2) + \log_4(x+5) = 3$
$\Rightarrow (x+2)(x+5) = 4^3$
$\Rightarrow x^2 + 7x + 10 = 64$
$\Rightarrow x^2 + 7x - 54 = 0$
$\Rightarrow x = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{-7\pm\sqrt{7^2-4\cdot1\cdot(-54)}}{2}$
$\Rightarrow x = \frac{-7\pm\sqrt{265}}{2}$
$\Rightarrow x = \frac{-7+\sqrt{265}}{2} \approx 4.639$ ✓ or $x = \frac{-7-\sqrt{265}}{2} \approx -11.639$ ✗
$\Rightarrow x = \frac{-7+\sqrt{265}}{2}$

Solving logarithmic equations - exercises

Solve the logarithmic equation.

12. $\log_2(x+7) - \log_2(x+3) = 5$

$\Rightarrow \log_2\left(\frac{x+7}{x+3}\right) = 5$

$\Rightarrow \frac{x+7}{x+3} = 2^5$

$\Rightarrow \frac{x+7}{x+3} = 32$

$\Rightarrow x+7 = 32\cdot(x+3)$

$\Rightarrow x+7 = 32x+96$

$\Rightarrow -31x = 89$

$\Rightarrow x = \frac{89}{-31} = -\frac{89}{31}$ ✓

Note that this is a solution, since both $\log_2(x+7)$ and $\log_2(x+3)$ are defined for $x = -\frac{89}{31}$, because:

$$x+7 = -\frac{89}{31} + 7 \approx 4.129 > 0,$$
$$x+3 = -\frac{89}{31} + 3 \approx 0.129 > 0.$$

13. $\log_3(x+5) - \log_3(x+2) = 4$

$\Rightarrow \log_3\left(\frac{x+5}{x+2}\right) = 4$

$\Rightarrow \frac{x+5}{x+2} = 3^4 = 81$

$\Rightarrow x+5 = 81\cdot(x+2)$

$\Rightarrow x+5 = 81x+162$

$\Rightarrow -80x = 157$

$\Rightarrow x = -\frac{157}{80}$ ✓

This is a solution, since $-\frac{157}{80} \approx -1.96$.

14. $\log_5(x+8) - \log_5(3-x) = 2$

$\Rightarrow \log_5\left(\frac{x+8}{3-x}\right) = 2$

$\Rightarrow \frac{x+8}{3-x} = 5^2 = 25$

$\Rightarrow x+8 = 25\cdot(3-x)$

$\Rightarrow x+8 = 75-25x$

$\Rightarrow 26x = 67$

$\Rightarrow x = \frac{67}{26}$ ✓

This is a solution, since $\frac{67}{26} \approx 2.58$, and so $x+8>0$ and $3-x>0$.