Exploring discontinuities and asymptotes

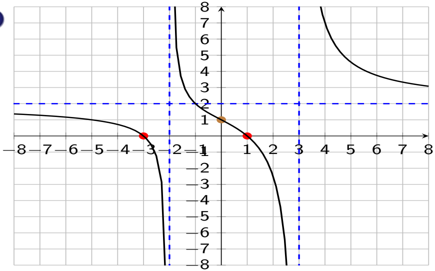
Lesson #11

MAT 1375 Precalculus

New York City College of Technology CUNY

Find the intercepts and the asymptotes of the rational function displayed below. What is its domain? What is its formula?

1



x-int.: $x=-3$, $x=1$ y-int.: $y=1$

VA: $x=-2, x=3$ HA: $y=2$

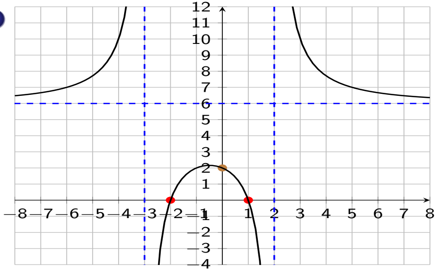
Domain: $D=(-\infty,-2)\cup(-2,3)\cup(3,\infty)$

$$f(x) = \frac{a\cdot(x-1)\cdot(x+3)}{(x+2)\cdot(x-3)}$$

$$f(0) = \frac{a\cdot(-1)\cdot 3}{2\cdot(-3)} = \frac{a}{2} \overset{!}{=} 1 \implies a=2$$

$$\implies f(x) = \frac{2\cdot(x-1)\cdot(x+3)}{(x+2)\cdot(x-3)}$$

2



x-int.: $x=-2$, $x=1$ y-int.: $y=2$

VA: $x=-3, x=2$ HA: $y=6$

Domain: $D=(-\infty,-3)\cup(-3,2)\cup(2,\infty)$

$$f(x) = \frac{a\cdot(x+2)\cdot(x-1)}{(x-2)\cdot(x+3)}$$

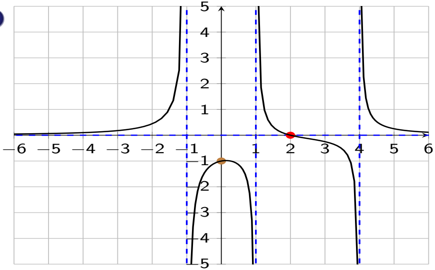
$$f(0) = \frac{a\cdot 2\cdot(-1)}{(-2)\cdot 3} = \frac{a}{3} \overset{!}{=} 2 \implies a=6$$

$$\implies f(x) = \frac{6\cdot(x+2)\cdot(x-1)}{(x-2)\cdot(x+3)}$$

Rational function via its graph - exercises

Find the intercepts and the asymptotes of the rational function displayed below. What is its domain? What is its formula?

3



x-int.: $x = 2$ y-int.: $y = -1$

VA: $x = -1, x = 1, x = 4$

HA: $y = 0$

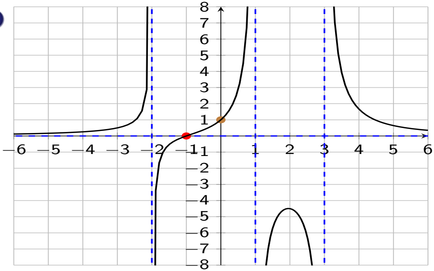
Domain: $D = (-\infty, -1) \cup (-1, 1) \cup (1, 4) \cup (4, \infty)$

$$f(x) = \frac{a\cdot(x-2)}{(x+1)\cdot(x-1)\cdot(x-4)}$$

$$f(0) = \frac{-2\cdot a}{4} = \frac{-a}{2} \overset{!}{=} -1 \implies a = 2$$

$$\implies f(x) = \frac{2\cdot(x-2)}{(x+1)\cdot(x-1)\cdot(x-4)}$$

4



x-int.: $x = -1$ y-int.: $y = 1$

VA: $x = -2, x = 1, x = 3$

HA: $y = 0$

Domain: $D = (-\infty, -2) \cup (-2, 1) \cup (1, 3) \cup (3, \infty)$

$$f(x) = \frac{a\cdot(x+1)}{(x-1)\cdot(x-3)\cdot(x+2)}$$

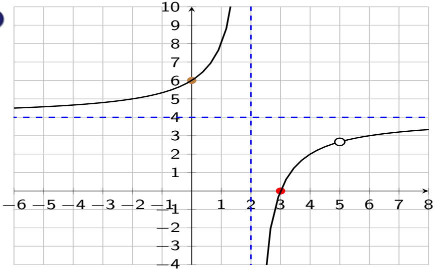
$$f(0) = \frac{a}{6} \overset{!}{=} 1 \implies a = 6$$

$$\implies f(x) = \frac{6\cdot(x+1)}{(x-1)\cdot(x-3)\cdot(x+2)}$$

Rational function via its graph - exercises

Find the intercepts and the asymptotes of the rational function displayed below. What is its domain? What is its formula?

5



x-int.: $x = 3$ y-int.: $y = 6$

VA: $x = 2$ HA: $y = 4$

Hole: $x = 5$

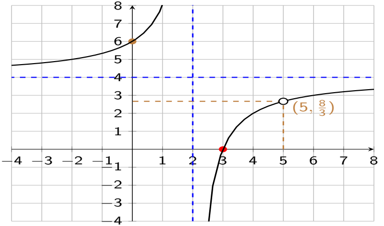
Domain: $D = (-\infty, 2) \cup (2, 5) \cup (5, \infty)$

$$f(x) = \frac{a\cdot(x-3)\cdot(x-5)}{(x-2)\cdot(x-5)}$$

$$f(0) = \frac{3a}{2} \overset{!}{=} 6 \implies a = \frac{2}{3} \cdot 6 = 4$$

$$\implies f(x) = \frac{4\cdot(x-3)\cdot(x-5)}{(x-2)\cdot(x-5)}$$

6 From previous exercise:



$f(x) = \frac{4\cdot(x-3)\cdot(x-5)}{(x-2)\cdot(x-5)}$ has hole at $x = 5$.

Q: What is the y-coordinate of the removable singularity?

A: For $x \neq 5$: $f(x) = \frac{4\cdot(x-3)}{(x-2)}$

So, as x approaches 5, $x \to 5$

$y = f(x)$ approaches

$$y \to \frac{4\cdot(5-3)}{(5-2)} = \frac{4\cdot 2}{3} = \frac{8}{3} \approx 2.667$$

$\implies$ The singularity is at $(x, y) = (5, \frac{8}{3})$.

Asymptotes of a rational function $f(x)$

Using a long division $\frac{f(x)}{q(x)}$, we can write $f(x) = g(x) + \frac{p(x)}{q(x)}$ for polynomials $g(x)$ and $p(x)$. When $\deg(p) < \deg(q)$, the rational function $\frac{p(x)}{q(x)}$ has a horizontal asymptote $y = 0$, and so:

$$\text{for } x \to \pm\infty: \quad f(x) \approx g(x)$$

1. Find the slant asymptote of

$$f(x) = \frac{3x^2-5x+2}{x-4}.$$

Solution: Use long division:

$$\begin{array}{r|rrr} & & 3x & +7 \\ \hline x-4 & 3x^2 & -5x & +2 \\ & -(3x^2 & -12x) & \\ \hline & & 7x & +2 \\ & & -(7x & -28) \\ \hline & & & 30 \end{array}$$

Thus: $f(x) = \frac{3x^2-5x+2}{x-4} = 3x + 7 + \frac{30}{x-4}$ has slant asymptote:

$$y = 3x + 7$$

2. Which polynomial does

$$f(x) = \frac{x^3+4x^2-5x+7}{x+2}$$

asymptotically approach?

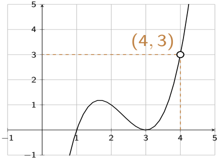
Solution:

$$\begin{array}{r|rrrr} & & x^2 & +2x & -9 \\ \hline x+2 & x^3 & +4x^2 & -5x & +7 \\ & -(x^3 & +2x^2) & & \\ \hline & & 2x^2 & -5x & \\ & & -(2x^2 & +4x) & \\ \hline & & & -9x & +7 \\ & & & -(-9x & -18) \\ \hline & & & & 25 \end{array}$$

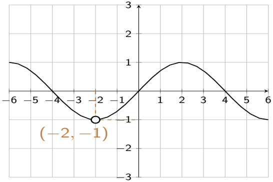
Therefore $f(x) = \frac{x^3+4x^2-5x+7}{x+2}$ approaches: $\quad y = x^2 + 2x - 9$

Function values approaching a discontinuity - exercises

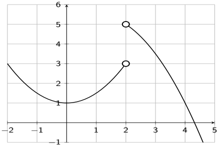
1. Assuming the following graph has a hole at (x, y) where x and y are integer values.



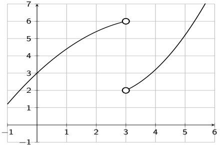
   As $x \to 4$, what does y approach?
   $y \to 3$

2. 

   As $x \to -2$, what does y approach?
   $y \to -1$

3. 

   When $x \to 2^+ \implies y \to 5$
   When $x \to 2^- \implies y \to 3$

4. 

   When $x \to 3^+ \implies y \to 2$
   When $x \to 3^- \implies y \to 6$

Limit

In calculus, this will be written as:

$$\lim_{x \to 3^+} f(x) = 2 \quad \text{and} \quad \lim_{x \to 3^-} f(x) = 6$$

Function values approaching a discontinuity using a formula - exercises

5. Let $f(x) = \frac{(x-2)}{2x \cdot |x-2|}$

When $x \to 2^+ \implies y \to 0.25$

When $x \to 2^- \implies y \to -0.25$

Use calculator to find values close to 2:

x	y
2.1	0.2380...
2.01	0.2487...
2.001	0.2498...

x	y
1.9	-0.2631...
1.99	-0.2512...
1.999	-0.2501...

$\lim_{x\to 2^+} f(x) = 0.25 \quad \lim_{x\to 2^-} f(x) = -0.25$

6. Let $f(x) = \frac{\sqrt{x}-1}{x-1}$

When $x \to 1^+ \implies y \to 0.5$

When $x \to 1^- \implies y \to 0.5$

Use calculator to find values close to 1:

x	y
1.1	0.4880...
1.01	0.4987...
1.001	0.4998...

x	y
0.9	0.5131...
0.99	0.5012...
0.999	0.5001...

We write: $\lim_{x\to 1^+} f(x) = 0.5$

$\lim_{x\to 1^-} f(x) = 0.5$

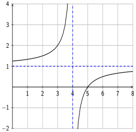
Thus: $\lim_{x\to 1} f(x) = 0.5$

7. Let $f(x) = \frac{x-5}{x-4}$

When $x \to 4^+ \implies y \to -\infty$

When $x \to 4^- \implies y \to +\infty$

Use calculator to find values close to 4:



x	y
4.1	-9
4.01	-99
4.001	-999

x	y
3.9	11
3.99	101
3.999	1001

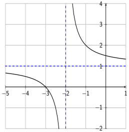
$\lim_{x\to 4^+} f(x) = -\infty \quad \lim_{x\to 4^-} f(x) = +\infty$

8. Let $f(x) = \frac{x+3}{x+2}$

When $x \to -2^+ \implies y \to +\infty$

When $x \to -2^- \implies y \to -\infty$

Use calculator to find values close to -2:



x	y
-1.9	11
-1.99	101
-1.999	1001

x	y
-2.1	-9
-2.01	-99
-2.01	-999

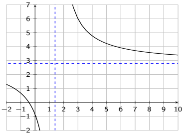
$\lim_{x\to -2^+} f(x) = +\infty \quad \lim_{x\to -2^-} f(x) = -\infty$

Approaching infinities - exercises

1. Let $f(x) = \frac{14x+6}{5x-7}$

   When $x \to +\infty \implies y \to 2.8$

   Use calculator to find larger and larger function values:



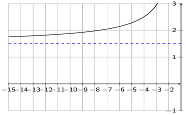
x	y
100	2.851...
1000	2.805...
10000	2.800...
100000	2.800...

$$\lim_{x \to +\infty} f(x) = 2.8 = \frac{14}{5}$$

2. Let $f(x) = \frac{3x^2-17x+4}{2x^2-6x-8}$

   When $x \to -\infty \implies y \to 1.5$

   Use calculator to find smaller and smaller function values:



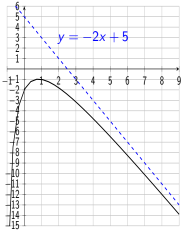
x	y
−100	1.539...
−1000	1.503...
−10000	1.500...
−100000	1.500...

$$\lim_{x \to -\infty} f(x) = 1.5 = \frac{3}{2}$$

3. Let $f(x) = \frac{-6x^2+7x-8}{3x+4}$

   When $x \to +\infty \implies y \to -\infty$

   Use calculator to find larger and larger function values:



x	y
100	-195.1...
1000	-1995.0...
10000	-19995.0...
100000	-199995.0...

$$\lim_{x \to +\infty} f(x) = -\infty$$

Note that we can compute the slant asymptote via a long division:

$$f(x) = -2x + 5 + \frac{-28}{3x+4}$$

$\implies$ slant asymptote is $y = -2x + 5$