

Rational functions

Lesson #10

MAT 1375 Precalculus

New York City College of Technology CUNY



Domain, vertical asymptotes, holes

Rational function, domain, VA, hole

A **rational function** is a function $f(x) = \frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomials.

Domain: $D = \{x : q(x) \neq 0\}$

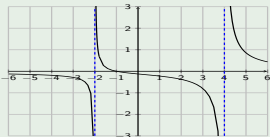
Vertical asymptotes (VA): $x = c$ for roots c of $q(x)$

Hole or removable singularity:

If the factors $(x - c)$ of $q(x)$ cancel with factors of $p(x)$, then $x = c$ is a hole.

Example

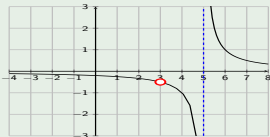
$$\text{Graph } f(x) = \frac{x+1}{(x+2)(x-4)}$$



Domain: $D = (-\infty, -2) \cup (-2, 4) \cup (4, \infty)$
vertical asymptotes: $x = -2, x = 4$

Example

$$\text{Graph } f(x) = \frac{x-3}{x^2-8x+15} = \frac{x-3}{(x-3)(x-5)}$$



Domain: $D = (-\infty, 3) \cup (3, 5) \cup (5, \infty)$
vertical asymptote: $x = 5$

$f(x) = \frac{x-3}{(x-3)(x-5)} = \frac{1}{x-5}$ for $x \neq 3$ only!
hole: $x = 3$

Domain, vertical asymptotes, holes - exercises

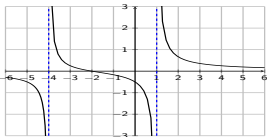
Find the domain, vertical asymptotes, and holes of the rational function.

1 $f(x) = \frac{x+2}{(x-1)(x+4)}$

Domain: $D = (-\infty, -4) \cup (-4, 1) \cup (1, \infty)$

VA: $x = 1, x = -4$

holes: no holes

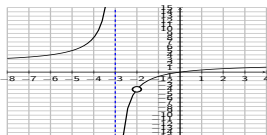


3 $f(x) = \frac{2x^2+4x}{x^2+5x+6} = \frac{2x(x+2)}{(x+2)(x+3)}$

Domain: $D = (-\infty, -3) \cup (-3, -2) \cup (-2, \infty)$

VA: $x = -3$

holes: $x = -2$

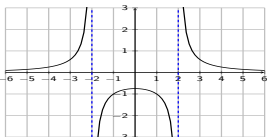


2 $f(x) = \frac{3}{x^2-4} = \frac{3}{(x+2)(x-2)}$

Domain: $D = (-\infty, -2) \cup (-2, 2) \cup (2, \infty)$

VA: $x = 2, x = -2$

holes: no holes

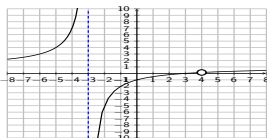


4 $f(x) = \frac{x^2-7x+12}{x^2-x-12} = \frac{(x-3)(x-4)}{(x+3)(x-4)}$

Domain: $D = (-\infty, -3) \cup (-3, 4) \cup (4, \infty)$

VA: $x = -3$

holes: $x = 4$



x-intercepts, y-intercept, horizontal asymptote

x-intercept(s)

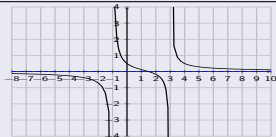
$$y = 0 \Rightarrow f(x) = 0 \Rightarrow \frac{p(x)}{q(x)} = 0 \Rightarrow p(x) = 0 \quad (\text{but } q(x) \neq 0)$$

y-intercept

$$x = 0 \Rightarrow y = f(0)$$

Horizontal asymptote of $f(x) = \frac{p(x)}{q(x)}$

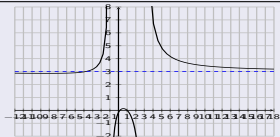
$$f(x) = \frac{2x-3}{2x^2-4x-6}$$



q has higher degree

$$\text{HA: } y = 0$$

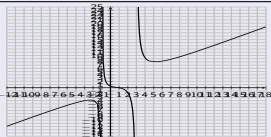
$$f(x) = \frac{6x^2-7x+1}{2x^2-4x-6}$$



p and q have same degree

$$f(x) = \frac{ax^n + \dots (\text{lower terms}) \dots}{bx^n + \dots (\text{lower terms}) \dots}$$
$$\text{HA: } y = \frac{a}{b}$$

$$f(x) = \frac{2x^3-3x^2+4x-3}{2x^2-4x-6}$$



p has higher degree

HA: there is no HA

x-intercepts, y-intercept, horizontal asymptotes - exercises

Find the horizontal asymptotes, x-intercepts, and y-intercept of the rational function.

1 $f(x) = \frac{4x^2 - 4}{8x^2 - 12x + 3}$

x-int.: $4x^2 - 4 = 0 \Rightarrow 4(x+1)(x-1) = 0 \Rightarrow x = \pm 1$

y-int.: $y = f(0) = \frac{-4}{3}$

HA: $y = \frac{4}{8} = \frac{1}{2}$

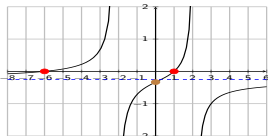


2 $f(x) = \frac{x^2 + 5x - 6}{18 - 4x^2}$

x-int.: $x^2 + 5x - 6 = 0 \Rightarrow (x-1)(x+6) = 0$
 $\Rightarrow x = 1, x = -6$

y-int.: $y = f(0) = \frac{-6}{18} = -\frac{1}{3}$

HA: $y = \frac{1}{-4}$

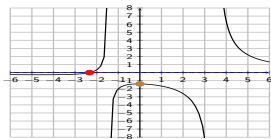


3 $f(x) = \frac{3x+7}{x^2-2x-5}$

x-int.: $3x + 7 = 0 \Rightarrow x = -\frac{7}{3}$

y-int.: $y = f(0) = \frac{7}{-5}$

HA: $y = 0$

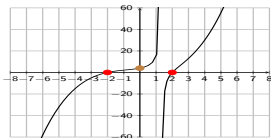


4 $f(x) = \frac{x^4 - 16}{3x - 4}$

x-int.: $x^4 - 16 = 0 \Rightarrow x^4 = 16 \Rightarrow x = \pm\sqrt[4]{16} = \pm 2$

y-int.: $y = f(0) = \frac{-16}{-4} = 4$

HA: no HA



Rational function - exercises

Find the domain, VA, holes, x-intercepts, y-intercept, and HA of the rational function.

$$1 \quad f(x) = \frac{2x-6}{x^2-3x-10} = \frac{2(x-3)}{(x-5)(x+2)}$$

Domain: $D = (-\infty, -2) \cup (-2, 5) \cup (5, \infty)$

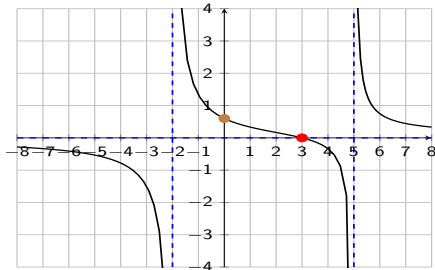
VA: $x = -2, x = 5$

holes: no holes

x-int.: $2x - 6 = 0 \Rightarrow 2(x - 3) = 0 \Rightarrow x = 3$

y-int.: $y = f(0) = \frac{-6}{-10} = \frac{3}{5}$

HA: $y = 0$



$$2 \quad f(x) = \frac{2x^2+2x-4}{x^2+x-12} = \frac{2(x-1)(x+2)}{(x-3)(x+4)}$$

Domain: $D = (-\infty, -4) \cup (-4, 3) \cup (3, \infty)$

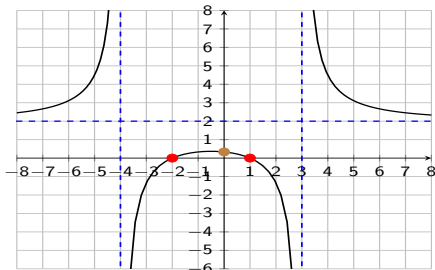
VA: $x = -4, x = 3$

holes: no holes

x-int.: $2(x - 1)(x + 2) = 0 \Rightarrow x = -2, x = 1$

y-int.: $y = f(0) = \frac{-4}{-12} = \frac{1}{3}$

HA: $y = \frac{2}{1} = 2$



Rational function - exercises

Find the domain, VA, holes, x-intercepts, y-intercept, and HA of the rational function.

$$3 \quad f(x) = \frac{x^2 - 4x - 5}{x^2 + 2x - 8} = \frac{(x-5)(x+1)}{(x-2)(x+4)}$$

Domain: $D = (-\infty, -4) \cup (-4, 2) \cup (2, \infty)$

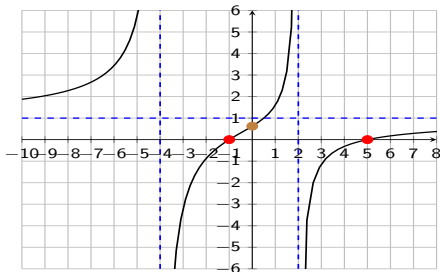
VA: $x = -4, x = 2$

holes: no holes

x-int.: $(x-5)(x+1) = 0 \Rightarrow x = 5, x = -1$

y-int.: $y = f(0) = \frac{-5}{-8} = \frac{5}{8}$

HA: $y = \frac{1}{1} = 1$



$$4 \quad f(x) = \frac{6x^2 - 24x + 18}{2x^2 - 4x - 6} = \frac{6(x-3)(x-1)}{2(x-3)(x+1)}$$

Domain: $D = (-\infty, -1) \cup (-1, 3) \cup (3, \infty)$

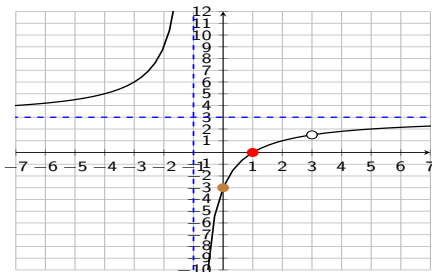
VA: $x = -1$

holes: $x = 3$

x-int.: $6x^2 - 24x + 18 = 0 \Rightarrow 6(x-3)(x-1) = 0$
 $\Rightarrow x = 3 \times$ $x = 1 \checkmark$

y-int.: $y = f(0) = \frac{18}{-6} = -3$

HA: $y = \frac{6}{2} = 3$



Rational function - exercises

Find the domain, VA, holes, x-intercepts, y-intercept, and HA of the rational function.

$$5 \quad f(x) = \frac{x^2 - 3x - 4}{x - 1} = \frac{(x-4)(x+1)}{x-1}$$

Domain: $D = (-\infty, 1) \cup (1, \infty)$

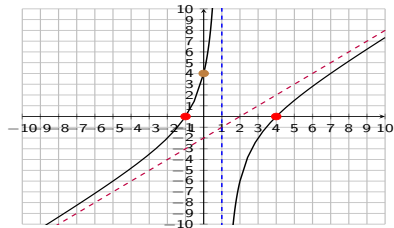
VA: $x = 1$

holes: no holes

x-int.: $(x-4)(x+1) = 0 \Rightarrow x = 4, x = -1$

y-int.: $y = f(0) = \frac{-4}{-1} = 4$

HA: no HA



Long division: $\frac{x^2 - 3x - 4}{x - 1} = x - 2 + \frac{-6}{x - 1}$
 \Rightarrow slant asymptote: $y = x - 2$

$$6 \quad f(x) = \frac{x+2}{x^3 - 4x^2} = \frac{x+2}{x^2(x-4)}$$

Domain: $D = (-\infty, 0) \cup (0, 4) \cup (4, \infty)$

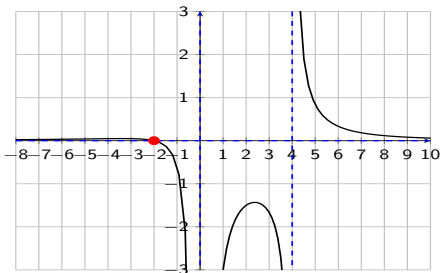
VA: $x = 0, x = 4$

holes: no holes

x-int.: $x + 2 = 0 \Rightarrow x = -2$

y-int.: $y = f(0) = \frac{2}{0}$ is undefined \times
 \Rightarrow no y-intercept

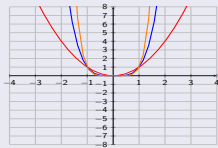
HA: $y = 0$



Graphs of $f(x) = x^n$ and $f(x) = \frac{1}{x^n}$

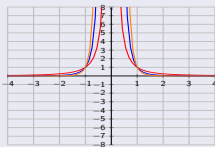
Explore the graphs $y = x^n$ and $y = \frac{1}{x^n}$ with desmos with n even or odd natural numbers.

$y = x^n$ with n even



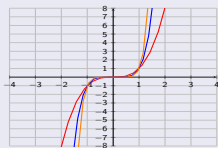
$$y = x^2$$
$$y = x^4$$
$$y = x^6$$

$y = \frac{1}{x^n}$ with n even



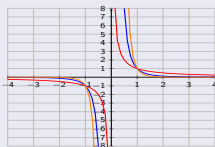
$$y = \frac{1}{x^2}$$
$$y = \frac{1}{x^4}$$
$$y = \frac{1}{x^6}$$

$y = x^n$ with n odd



$$y = x^3$$
$$y = x^5$$
$$y = x^7$$

$y = \frac{1}{x^n}$ with n odd



$$y = \frac{1}{x^3}$$
$$y = \frac{1}{x^5}$$
$$y = \frac{1}{x^7}$$

