Roots of polynomials

Lesson #9

MAT 1375 Precalculus

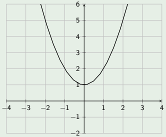
New York City College of Technology CUNY

Theorem (The fundamental theorem of algebra)

Every polynomial, which is not constant, has a root.

Example

$f(x) = x^2 + 1$



Where are the roots???
Try the quadratic formula:

$$x^2 + 0x + 1 = 0$$

$$\Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-0 \pm \sqrt{0^2 - 4 \cdot 1 \cdot 1}}{2 \cdot 1} = \frac{\pm\sqrt{-4}}{2}$$
$$= \frac{\pm\sqrt{4}\sqrt{-1}}{2} = \frac{\pm 2i}{2} = \pm i$$

f has roots i and $-i$
f has factors
$(x - i)$ and $(x + i)$
Check:
$(x + i) \cdot (x - i) =$
$= x^2 + ix - ix - i^2$
$= x^2 - (-1) = x^2 + 1$ ✓

Note: Every polynomial has a root, but it may be a **complex root**!

Fact: If the polynomial f with real coefficients has the complex root $c = a + bi$, then the complex conjugate $\bar{c} = a - bi$ is also a root!

Factor polynomials completely

Start with a polynomial $f(x)$, say of degree n.

1) Find root c_1 of f and factor (using long division): $f(x) = (x - c_1) \cdot g(x)$
2) Find root c_2 of g and factor (using long division): $f(x) = (x - c_1) \cdot (x - c_2) \cdot h(x)$

$\vdots$

n) After finding n roots $c_1, \ldots, c_n$, we can write $f(x)$ as:

$$f(x) = a \cdot (x - c_1) \cdot (x - c_2) \cdot \ldots \cdot (x - c_n)$$

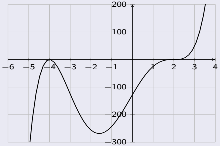
There is one more piece of information: the leading coefficient a

Note: We call $c_1, \ldots, c_n$ the **roots** of f, and $(x - c_1), \ldots, (x - c_n)$ the **factors** of f.

Note: A polynomial $f(x) = ax^n + bx^{n-1} + \ldots$ of degree n has at most n roots $c_1, \ldots, c_n$.

Note: A root may appear multiple times. How often the root appears is called the **multiplicity** of the root.

Example:
$f(x) = (x-2)^3 \cdot (x+4)^2$
root 2 has multiplicity 3
root -4 has multiplicity 2



Find a polynomial that fits the given data.

1. degree 3
   roots 2, -3, 1
   $$\begin{aligned} f(x) &= (x-2)(x+3)(x-1) \\ &= (x^2-2x+3x-6)(x-1) \\ &= (x^2+x-6)(x-1) \\ &= x^3-x^2+x^2-x-6x+6 \\ &= x^3-7x+6 \end{aligned}$$
2. degree 3
   roots -1, $+1$, -4
   $$\begin{aligned} f(x) &= (x+1)(x-1)(x+4) \\ &= (x^2-x+x-1)(x+4) \\ &= (x^2-1)(x+4) \\ &= x^3+4x^2-x-4 \end{aligned}$$
3. degree 3
   roots -1, $+1$, -4
   passes through $(2, 36)$
   $f(x) = a \cdot (x+1)(x-1)(x+4)$ and $f(2) = 36$
   $f(2) = a \cdot (2+1)(2-1)(2+4) = a \cdot 3 \cdot 1 \cdot 6 = 18a$
   Therefore, $18a = 36 \quad \Rightarrow a = 2$
   $$\begin{aligned} f(x) &= 2 \cdot (x+1)(x-1)(x+4) \\ &= 2 \cdot (x^3+4x^2-x-4) \\ &= 2x^3+8x^2-2x-8 \end{aligned}$$
4. degree 4
   roots 0, -1, 2, -2
   passes through $(1, 24)$
   $f(x) = a \cdot x(x+1)(x-2)(x+2)$ and $f(1) = 24$
   $f(1) = a \cdot 1 \cdot 2 \cdot (-1) \cdot 3 = -6a$
   Therefore, $-6a = 24 \quad \Rightarrow a = -4$
   $$\begin{aligned} f(x) &= (-4) \cdot x(x+1) \cdot (x-2)(x+2) \\ &= (-4x^2-4x) \cdot (x^2-4) \\ &= -4x^4-4x^3+16x^2+16x \end{aligned}$$

Find a polynomial that fits the given data.

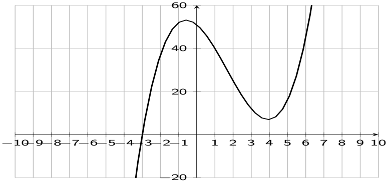
5. degree 2
   roots $2+3i$, $2-3i$
   $$\begin{aligned} f(x) &= (x-(2+3i))\cdot(x-(2-3i)) \\ &= (x-2-3i)\cdot(x-2+3i) \\ &= x^2-2x+3xi-2x+4-6i \\ &\qquad -3xi+6i-9i^2 \\ &= x^2-4x+4-9\cdot(-1) \\ &= x^2-4x+4+9 \\ &= x^2-4x+13 \end{aligned}$$

6. degree 2
   roots $5+i$, $5-i$
   $$\begin{aligned} f(x) &= (x-(5+i))\cdot(x-(5-i)) \\ &= (x-5-i)\cdot(x-5+i) \\ &= ((x-5)-i)\cdot((x-5)+i) \\ &= (x-5)^2-i^2 \\ &= x^2-10x+25-(-1) \\ &= x^2-10x+26 \end{aligned}$$

7. degree 3
   roots 3, $4+2i$, $4-2i$
   $$\begin{aligned} f(x) &= (x-3)\cdot(x-(4+2i))\cdot(x-(4-2i)) \\ &= (x-3)\cdot(x-4-2i)\cdot(x-4+2i) \\ &= (x-3)\cdot(x^2-4x+2xi \\ &\qquad -4x+16-8i-2xi+8i-4i^2) \\ &= (x-3)\cdot(x^2-8x+16+4) \\ &= (x-3)\cdot(x^2-8x+20) \\ &= x^3-8x^2+20x-3x^2+24x-60 \\ &= x^3-11x^2+44x-60 \end{aligned}$$

Factor completely - exercises

1. Find all roots and find all factors of $f(x) = x^3 - 5x^2 - 7x + 51$
   - Step 1: Find a root c of $f(x)$ with a graphing calculator



Observe: -3 is a root of $f(x)$.
Therefore: divide $f(x)$ by $(x+3)$

   - Step 2: Divide $f(x)$ by $(x - c)$

$$\begin{array}{r|rrrr}
 & & x^2 & -8x & +17 \\ \hline
x+3 & x^3 & -5x^2 & -7x & +51 \\
 & -(x^3 & +3x^2) & & \\ \hline
 & & -8x^2 & -7x & \\
 & & -(-8x^2 & -24x) & \\ \hline
 & & & 17x & +51 \\
 & & & -(17x & +51) \\ \hline
 & & & & 0 \checkmark
\end{array}$$

Thus:
$x^3 - 5x^2 - 7x + 51 = (x^2 - 8x + 17) \cdot (x+3)$

   - Step 3: Continue factoring; use factoring or the quadratic formula if possible:

$x^2 - 8x + 17 = 0 \quad \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-8) \pm \sqrt{(-8)^2 - 4 \cdot 1 \cdot 17}}{2} = \frac{8 \pm \sqrt{-4}}{2} = \frac{8 \pm 2i}{2} = 4 \pm i$

$\Rightarrow x^3 + 5x^2 - x - 14 = (x+3)(x^2 - 8x + 17) = (x+3) \cdot (x - (4+i)) \cdot (x - (4-i))$

Factor completely - exercises

2. a) Find a real number C so that the polynomial

$$f(x) = 4x^3 - 12x^2 + 5x + C$$

has a root at $x = 2$.

b) For this C, find all remaining roots of the polynomial algebraically and factor the polynomial completely.

- 2 is a root of $f(x)$. Therefore, divide $4x^3 - 12x^2 + 5x + C$ by $(x - 2)$:

$$\begin{array}{r|rrrr}
 & 4x^2 & -4x & -3 & \\
\hline
x-2 & 4x^3 & -12x^2 & +5x & +C \\
 & -(4x^3 & -8x^2) & & \\
\cline{2-3}
 & & -4x^2 & +5x & \\
 & & -(-4x^2 & +8x) & \\
\cline{3-4}
 & & & -3x & +C \\
 & & & -(-3x & +6) \\
\cline{4-5}
 & & & & C-6
\end{array}$$

- We need a remainder of zero, that is, we need that $C - 6 = 0$. This gives $C = 6$, and so

$$\Longrightarrow \quad f(x) = 4x^3 - 12x^2 + 5x + 6$$

- Continue factoring (use quadratic formula if necessary): $\quad 4x^2 - 4x - 3 = 0$

$$\Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \cdot 4 \cdot (-3)}}{8} = \frac{4 \pm \sqrt{64}}{8} = \frac{4 \pm 8}{8} = \frac{4 \cdot (1 \pm 2)}{8} \; \frac{1 \pm 2}{2}$$

$$\Rightarrow x = \frac{1+2}{2} = \frac{3}{2} \qquad \text{or} \qquad x = \frac{1-2}{2} = \frac{-1}{2}$$

$$\Rightarrow 4x^3 - 12x^2 + 5x + 6 = (x-2)(4x^2 - 4x - 3) = 4 \cdot (x-2) \cdot \left(x - \frac{3}{2}\right) \cdot \left(x + \frac{1}{2}\right)$$