Graphing polynomials

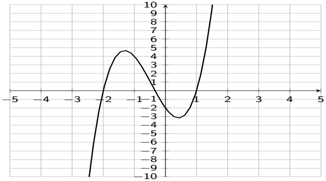
Lesson #8

MAT 1375 Precalculus

New York City College of Technology CUNY

Factor completely - exercise

1. Find all roots and find all factors of $f(x) = 3x^3 + 4x^2 - 5x - 2$
   - Step 1: Find a root c of $f(x)$ with a graphing calculator



Observe: 1 and -2 are roots of $f(x)$.
Therefore: divide $f(x)$ by either $(x+2)$ or $(x-1)$ (choose one)

   - Step 2: Divide $f(x)$ by $(x - c)$

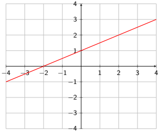
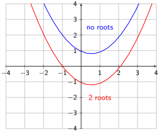
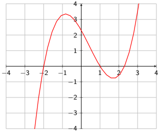
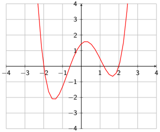
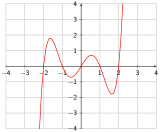
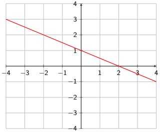
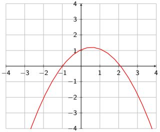
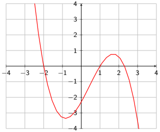
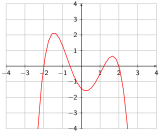
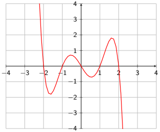
$$
\begin{array}{r|rrrr}
 & 3x^2 & -2x & -1 & \\
\hline
x+2 & 3x^3 & +4x^2 & -5x & -2 \\
 & -(3x^3 & +6x^2) & & \\
\hline
 & & -2x^2 & -5x & \\
 & & -(-2x^2 & -4x) & \\
\hline
 & & & -x & -2 \\
 & & & -(-x & -2) \\
\hline
 & & & & 0 \checkmark
\end{array}
$$

Thus:
$3x^3 + 4x^2 - 5x - 2 = (3x^2 - 2x - 1) \cdot (x + 2)$

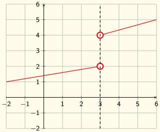
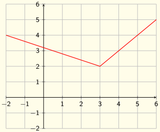
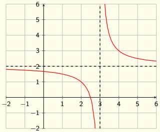
   - Step 3: Continue factoring; use factoring or the quadratic formula if possible:
     $3x^2 - 2x - 1 = (3x + 1)(x - 1)$
     $\Rightarrow x^3 + 5x^2 - x - 14 = (3x^2 - 2x - 1)(x + 2) = (3x + 1)(x - 1)(x + 2) = 3\left(x + \frac{1}{3}\right)(x - 1)(x + 2)$

   - Solution: Factors: $\left(x + \frac{1}{3}\right)$, $(x - 1)$, $(x + 2)$, Roots: $-\frac{1}{3}$, 1, -2

Graphs of polynomials

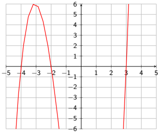
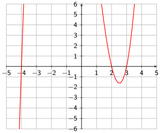
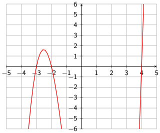
degree 1	degree 2	degree 3	degree 4	degree 5
$y = ax + b$	$y = ax^2 + bx + c$	$y = ax^3 + bx^2 + cx + d$	$y = ax^4 + bx^3 + cx^2 + dx + e$	$y = ax^5 + bx^4 + cx^3 + dx^2 + ex + f$
				
$a > 0$	$a > 0$	$a > 0$	$a > 0$	$a > 0$
				
$a < 0$	$a < 0$	$a < 0$	$a < 0$	$a < 0$
1 real root	at most 2 real root	at most 3 real root	at most 4 real root	at most 5 real root
no max. or min.	1 max. or min.	at most 2 max. or min.	at most 3 max. or min.	at most 4 max. or min.

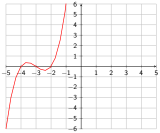
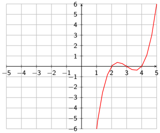
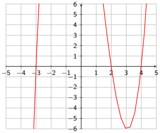
Polynomials do ***NOT*** have the following:

discontinuity:  corner:  asymptotes: 

1. Match the functions with their graphs:

(C)	$f(x) = (x-2)(x-3)(x+4)$	(D)	$h(x) = (x-2)(x-3)(x-4)$
(B)	$g(x) = (x+2)(x+3)(x+4)$	(A)	$k(x) = (x+2)(x-3)(x+4)$

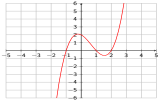
(A)  (C)  (E) 

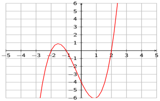
(B)  (D)  (F) 

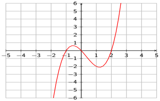
Notation: $f(x) = (x-2)(x-3)(x+4)$ has **factors**: $(x-2)$ and $(x-3)$ and $(x+4)$
and has **roots**: 2 and 3 and -4.

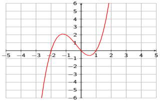
Match the functions with their graphs:

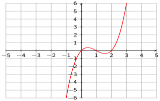
2. (E) $f(x) = (x+1)(x-2)x$ (F) $h(x) = (x-1)(x+1)x$
   (D) $g(x) = (x-2)(x-1)x$ (A) $k(x) = (x+1)(x-2)(x-1)$

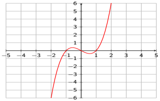
(A) 

(C) 

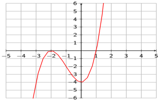
(E) 

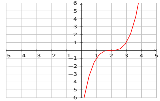
(B) 

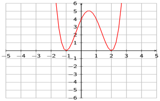
(D) 

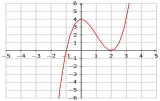
(F) 

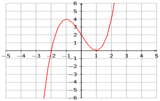
3. (D) $f(x) = (x-1)^2(x+2)$ (B) $h(x) = (x-2)^2(x+1)$
   (C) $g(x) = (x-2)^3$ (E) $k(x) = (x+1)^2(x-2)^2$

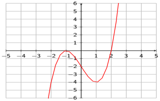
(A) 

(C) 

(E) 

(B) 

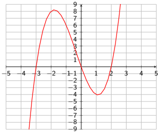
(D) 

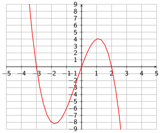
(F) 

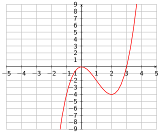
Match the functions with their graphs:

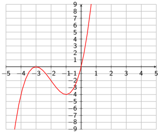
4. $f(x) = x^3 - 5x^2 + 6x$ $\quad$ $h(x) = x^3 + x^2 - 6x$

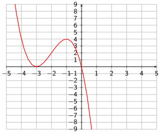
$g(x) = -x^3 - 6x^2 - 9x$ $\quad$ $k(x) = -x^3 - x^2 + 6x$

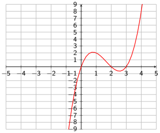
(A) 

(C) 

(E) 

(B) 

(D) 

(F) 

Factor the functions f, g, h, k:

(F) $\quad f(x) = x^3 - 5x^2 + 6x = x(x^2 - 5x + 6) = x(x-2)(x-3)$

(D) $\quad g(x) = -x^3 - 6x^2 - 9x = -x(x^2 + 6x + 9) = -x(x+3)(x+3)$

(A) $\quad h(x) = x^3 + x^2 - 6x = x(x^2 + x - 6) = x(x-2)(x+3)$

(C) $\quad k(x) = -x^3 - x^2 + 6x = -x(x^2 + x - 6) = -x(x-2)(x+3)$

Integer roots of polynomials

Example

- The polynomial $f(x) = (x+3) \cdot (2x^2 + 4x + 5)$ has the *integer* root -3.
- Multiply this: $f(x) = 2x^3 + 6x^2 + 4x^2 + 12x + 5x + 15 = 2x^3 + 10x^2 + 17x + 15$.
- Indeed all integer roots of $f(x)$ have to be factors of the last coefficient 15.
- The possibilities are:

$$+1, -1, +3, -3, +5, -5, +15, -15$$

Integer roots

Assume that $f(x) = a_n \cdot x^n + a_{n-1} \cdot x^{n-1} + \cdots + a_2 \cdot x^2 + a_1 \cdot x + a_0$ is a polynomial. If all coefficients $a_0, a_1, \ldots, a_n$ are integers, then any integer root of f is a factor of a_0.

List all possible integer roots of the polynomial according to the above theorem.

1. $f(x) = x^3 + 2x^2 - 4x + 7$
   Answer:
   $+1, -1, +7, -7$
2. $f(x) = x^3 - x^2 + 12$
   Answer:
   $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$
3. $f(x) = 3x^5 + 23x^4 - 58x - 8$
   Answer:
   $\pm 1, \pm 2, \pm 4, \pm 8$

Rational roots of polynomials

Example

- The polynomial $f(x) = (7x + 3) \cdot (2x^2 + 4x + 5)$ has the *rational* root $-\frac{3}{7}$.
- Multiply this: $f(x) = 14x^3 + 6x^2 + 28x^2 + 12x + 35x + 15 = 14x^3 + 34x^2 + 47x + 15$.
- Indeed for all rational roots $\frac{p}{q}$ of $f(x)$, which are completely reduced, p has to be a factor of the last coefficient 15, and q has to be a factor of the first coefficient 14.
- The possibilities are:

$$\pm\frac{1}{1}, \pm\frac{3}{1}, \pm\frac{5}{1}, \pm\frac{15}{1}, \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \pm\frac{15}{2}, \pm\frac{1}{7}, \pm\frac{3}{7}, \pm\frac{5}{7}, \pm\frac{15}{7}, \pm\frac{1}{14}, \pm\frac{3}{14}, \pm\frac{5}{14}, \pm\frac{15}{14}$$

Rational roots

Assume that $f(x) = a_n \cdot x^n + a_{n-1} \cdot x^{n-1} + \cdots + a_2 \cdot x^2 + a_1 \cdot x + a_0$ is a polynomial. If all coefficients $a_0, a_1, \ldots, a_n$ are integers, then for any rational root $\frac{p}{q}$ of f which is completely reduced, p is a factor of a_0, and q is a factor of a_n.

List all possible rational roots of the polynomial according to the above theorem.

1. $f(x) = x^3 + 2x^2 - 4x + 7$
   Answer:
   $+1, -1, +7, -7$
2. $f(x) = 3x^5 + 23x^4 - 58x - 8$
   Answer:
   $\pm 1, \pm 2, \pm 4, \pm 8, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{4}{3}, \pm\frac{8}{3}$