Dividing polynomials

Lesson #7

MAT 1375 Precalculus

New York City College of Technology CUNY

- Recall long division of integers.
  Divide: $\frac{3571}{11} = \frac{\text{dividend}}{\text{divisor}}$

$$
\begin{array}{r|l}
 & 324 \quad = \text{quotient} \\
\hline
11 & 3571 \\
 & \underline{-33} \\
 & 27 \\
 & \underline{-22} \\
 & 51 \\
 & \underline{-44} \\
 & 7 \quad = \text{remainder}
\end{array}
$$

Answer:

$$\frac{3571}{11} = 324 + \frac{7}{11}$$

Or rewrite without fractions:

$$3571 = 324 \cdot 11 + 7$$

- Divide polynomials: $\frac{x^3+5x^2+4x+2}{x+3}$

$$
\begin{array}{r|rrrr}
 & x^2 & +2x & -2 & \\
\hline
x+3 & x^3 & +5x^2 & +4x & +2 \\
 & -(x^3 & +3x^2) & & \\
\hline
 & & 2x^2 & +4x & \\
 & & -(2x^2 & +6x) & \\
\hline
 & & & -2x & +2 \\
 & & & -(-2x & -6) \\
\hline
 & & & & 8
\end{array}
$$

Answer:

$$\frac{x^3+5x^2+4x+2}{x+3} = x^2+2x-2+\frac{8}{x+3}$$

Or:

$$x^3+5x^2+4x+2 = (x^2+2x-2)\cdot(x+3)+8$$

Long division - exercises

1. Divide: $\frac{x^3-3x^2-4x+9}{x+2}$

$$
\begin{array}{r|rrrr}
 & x^2 & -5x & +6 & \\
\hline
x+2 & x^3 & -3x^2 & -4x & +9 \\
 & -(x^3 & +2x^2) & & \\
\hline
 & & -5x^2 & -4x & \\
 & & -(-5x^2 & -10x) & \\
\hline
 & & & 6x & +9 \\
 & & & -(6x & +12) \\
\hline
 & & & & -3
\end{array}
$$

Answer:

$$\frac{x^3-3x^2-4x+9}{x+2} = x^2-5x+6+\frac{-3}{x+2}$$

Or:

$$x^3-3x^2-4x+9 = (x^2-5x+6)\cdot(x+2)-3$$

2. Divide: $\frac{x^3-5x^2-2x+16}{x-4}$

$$
\begin{array}{r|rrrr}
 & x^2 & -x & -6 & \\
\hline
x-4 & x^3 & -5x^2 & -2x & +16 \\
 & -(x^3 & -4x^2) & & \\
\hline
 & & -x^2 & -2x & \\
 & & -(-x^2 & +4x) & \\
\hline
 & & & -6x & +16 \\
 & & & -(-6x & +24) \\
\hline
 & & & & -8
\end{array}
$$

Answer:

$$\frac{x^3-5x^2-2x+16}{x-4} = x^2-x-6-\frac{8}{x-4}$$

Or:

$$x^3-5x^2-2x+16 = (x^2-x-6)\cdot(x-4)-8$$

Long division - exercises

3. Divide: $\frac{6x^3+16x^2-7x+4}{3x+2}$

$$
\begin{array}{r|rrrr}
 & 2x^2 & +4x & -5 & \\
\hline
3x+2 & 6x^3 & +16x^2 & -7x & +4 \\
 & -(6x^3 & +4x^2) & & \\
\hline
 & & 12x^2 & -7x & \\
 & & -(12x^2 & +8x) & \\
\hline
 & & & -15x & +4 \\
 & & & -(-15x & -10) \\
\hline
 & & & & 14
\end{array}
$$

Answer:

$$\frac{6x^3+16x^2-7x+4}{3x+2} = 2x^2+4x-5+\frac{14}{3x+2}$$

Or:

$$6x^3+16x^2-7x+4 = (2x^2+4x-5)\cdot(3x+2)+14$$

4. Divide: $\frac{x^3+2x^2-5x+3}{x-2}$

$$
\begin{array}{r|rrrr}
 & x^2 & +4x & +3 & \\
\hline
x-2 & x^3 & +2x^2 & -5x & +3 \\
 & -(x^3 & -2x^2) & & \\
\hline
 & & 4x^2 & -5x & \\
 & & -(4x^2 & -8x) & \\
\hline
 & & & 3x & +3 \\
 & & & -(3x & -6) \\
\hline
 & & & & 9
\end{array}
$$

Answer:

$$\frac{x^3+2x^2-5x+3}{x-2} = x^2+4x+3+\frac{9}{x-2}$$

Or:

$$x^3+2x^2-5x+3 = (x^2+4x+3)\cdot(x-2)+9$$

From the last exercise:

$$\underbrace{x^3 + 2x^2 - 5x + 3}_{=f(x)} = (\underbrace{x^2 + 4x + 3}_{=q(x)}) \cdot (x-2) + 9$$

$$\Rightarrow f(x) = q(x) \cdot (x-2) + 9$$

$$\Rightarrow f(2) = q(2) \cdot (\underbrace{2-2}_{=0}) + 9 = 9$$

Theorem (Remainder theorem)

Divide $f(x)$ by $(x-c)$ gives $f(x) = q(x) \cdot (x-c) + R$, where the remainder R is given by

$$f(c) = R$$

Note that $R = 0$, that is $f(c) = 0$, exactly when $f(x) = q(x) \cdot (x-c)$, that is we succeeded in factoring $f(x)$ with $(x-c)$ being one factor.

Theorem (Factor theorem)

$(x-c)$ is a factor of $f(x)$ exactly when $f(c) = 0$, that is exactly when c is a root of f.

Find the remainder using the remainder theorem.

1. $\frac{x^3 - 4x^2 + 2x + 8}{x-3}$ $\quad c = 3$

$$\begin{aligned} R = f(3) &= 3^3 - 4 \cdot 3^2 + 2 \cdot 3 + 8 \\ &= 27 - 4 \cdot 9 + 6 + 8 \\ &= 27 - 36 + 6 + 8 \\ &= 5 \end{aligned}$$

2. $\frac{x^3 + 6x^2 + 11x + 6}{x+2}$ $\quad c = -2$

$$\begin{aligned} R &= f(-2) \\ &= (-2)^3 + 6 \cdot (-2)^2 + 11 \cdot (-2) + 6 \\ &= -8 + 6 \cdot 4 - 22 + 6 \\ &= -8 + 24 - 22 + 6 \\ &= 0 \end{aligned}$$

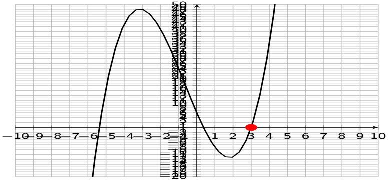
Therefore: $(x+2)$ is a factor of $x^3 + 6x^2 + 11x + 6$:

$$x^3 + 6x^2 + 11x + 6 = q(x) \cdot (x+2)$$

Find $q(x)$ via long division.

Factoring completely - exercises

1. Factor completely: $f(x) = x^3 + 2x^2 - 17x + 6$
   - Step 1: Find a root c of $f(x)$ with a graphing calculator



Observe: 3 is a root of $f(x)$.
Therefore: divide $f(x)$ by $(x - 3)$

   - Step 2: Divide $f(x)$ by $(x - c)$

$$
\begin{array}{r|rrrr}
 & x^2 & +5x & -2 & \\
\hline
x-3 & x^3 & +2x^2 & -17x & +6 \\
 & -(x^3 & -3x^2) & & \\
\hline
 & & 5x^2 & -17x & \\
 & & -(5x^2 & -15x) & \\
\hline
 & & & -2x & +6 \\
 & & & -(-2x & +6) \\
\hline
 & & & & 0 \checkmark
\end{array}
$$

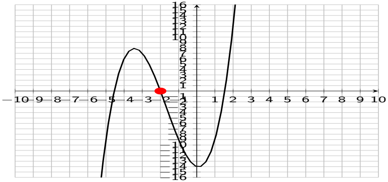
Thus:
$x^3 + 2x^2 - 17x + 6 = (x^2 + 5x - 2) \cdot (x - 3)$

   - Step 3: Continue factoring; use factoring or the quadratic formula if possible:

$$x^2 + 5x - 2 = 0 \quad \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-5 \pm \sqrt{5^2 - 4 \cdot 1 \cdot (-2)}}{2} = \frac{-5 \pm \sqrt{33}}{2}$$

$$\Rightarrow x^3 + 2x^2 - 17x + 6 = (x-3)(x^2+5x-2) = (x-3)\left(x - \frac{-5+\sqrt{33}}{2}\right)\left(x - \frac{-5-\sqrt{33}}{2}\right)$$

2. Factor completely: $f(x) = x^3 + 5x^2 - x - 14$
   - Step 1: Find a root c of $f(x)$ with a graphing calculator



Observe: -2 is a root of $f(x)$.
Therefore: divide $f(x)$ by $(x+2)$

   - Step 2: Divide $f(x)$ by $(x - c)$

$$\begin{array}{r|rrrr}
 & x^2 & +3x & -7 & \\
\hline
x+2 & x^3 & +5x^2 & -x & -14 \\
 & -(x^3 & +2x^2) & & \\
\hline
 & & 3x^2 & -x & \\
 & & -(3x^2 & +6x) & \\
\hline
 & & & -7x & -14 \\
 & & & -(-7x & -14) \\
\hline
 & & & & 0 \checkmark
\end{array}$$

Thus:
$x^3 + 5x^2 - x - 14 = (x^2 + 3x - 7) \cdot (x + 2)$

   - Step 3: Continue factoring; use factoring or the quadratic formula if possible:

$$x^2 + 3x - 7 = 0 \quad \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-3 \pm \sqrt{3^2 - 4 \cdot 1 \cdot (-7)}}{2} = \frac{-3 \pm \sqrt{37}}{2}$$

$$\Rightarrow x^3 + 5x^2 - x - 14 = (x+2)(x^2 + 3x - 7) = (x+2)\left(x - \frac{-3 + \sqrt{37}}{2}\right)\left(x - \frac{-3 - \sqrt{37}}{2}\right)$$