The inverse of a function
Lesson #6

MAT 1375 Precalculus

New York City College of Technology CUNY

Definition

A function $y = f(x)$ is called *one-to-one* if

two different inputs $x_1 \neq x_2$ always give two different outputs $f(x_1) \neq f(x_2)$.

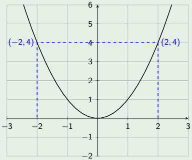
Example

Question: Is $f(x) = x^2$ one-to-one?

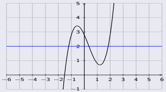
$f(2) = 4$
$f(-2) = 4$

No! ✗

f is not one-to-one!

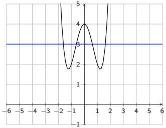
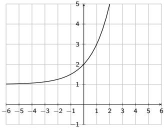
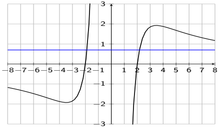


Horizontal line test



If there is a horizontal line, which intersects the graph of a function in more than one point, then the function is not one-to-one.

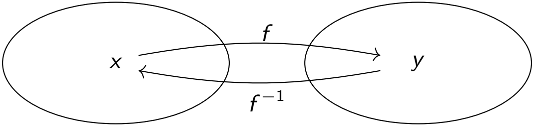
Is the function one-to-one?

1.  No!
2.  Yes!
3.  No!

The inverse function

If f is one-to-one, then every output y comes from exactly one input x.

Domain D_f $=$Range $R_{f^{-1}}$



Range R_f $=$Domain $D_{f^{-1}}$

Definition

Let $y = f(x)$ be a one-to-one function.
Define the *inverse function of* f to be the function f^{-1} which maps

$$y = f(x) \quad \Leftrightarrow \quad x = f^{-1}(y)$$

f^{-1} has domain $D_{f^{-1}} = R_f$, and has range $R_{f^{-1}} = D_f$.

Note: Composing f and f^{-1} gives back the original input:

$$f^{-1}(f(x)) = x \quad \text{and} \quad f(f^{-1}(x)) = x$$

Even better:
A function g is the inverse of f, that is $g = f^{-1}$, precisely when $D_g = R_f$, $R_g = D_f$, and

$$g(f(x)) = x \quad \text{and} \quad f(g(x)) = x$$

Steps for finding the inverse function

Step 1: Write $y = f(x)$ and switch the xs and ys.
Step 2: Solve for y.

Find the inverse function.

1. $f(x) = 3x - 7$
   Write this as $y = 3x - 7$
   Step 1: $x = 3y - 7$
   Step 2: $x + 7 = 3y$
   $\Rightarrow \frac{x+7}{3} = y$
   $\Rightarrow f^{-1}(x) = \frac{x+7}{3}$
2. $f(x) = 6x + 4$
   Write this as $y = 6x + 4$
   Step 1: $x = 6y + 4$
   Step 2: $x - 4 = 6y$
   $\Rightarrow \frac{x-4}{6} = y$
   $\Rightarrow f^{-1}(x) = \frac{x-4}{6}$
3. $f(x) = \sqrt{2x - 5}$
   Write this as
   $y = \sqrt{2x - 5}$
   Step 1: $x = \sqrt{2y - 5}$
   Step 2: $x^2 = 2y - 5$
   $\Rightarrow x^2 + 5 = 2y$
   $\Rightarrow \frac{x^2+5}{2} = y$
   $\Rightarrow f^{-1}(x) = \frac{x^2+5}{2}$

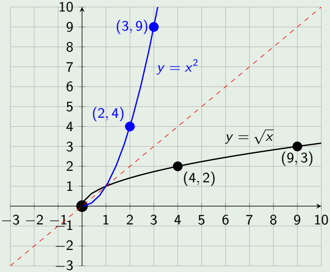
Inverse function - exercises

Find the inverse function.

4. $f(x) = \frac{2x+3}{4x+5}$
   Write this as $y = \frac{2x+3}{4x+5}$
   Step 1: $x = \frac{2y+3}{4y+5}$
   Step 2: $x(4y+5) = (2y+3)$
   $\Rightarrow 4xy + 5x = 2y + 3$
   $\Rightarrow 4xy - 2y = 3 - 5x$
   $\Rightarrow y(4x-2) = 3 - 5x$
   $\Rightarrow y = \frac{3-5x}{4x-2}$
   $\Rightarrow f^{-1}(x) = \frac{3-5x}{4x-2}$
5. $f(x) = \frac{7x-4}{9x-2}$
   Write this as $y = \frac{7x-4}{9x-2}$
   Step 1: $x = \frac{7y-4}{9y-2}$
   Step 2: $x(9y-2) = 7y - 4$
   $\Rightarrow 9xy - 2x = 7y - 4$
   $\Rightarrow 9xy - 7y = 2x - 4$
   $\Rightarrow y(9x-7) = 2x - 4$
   $\Rightarrow y = \frac{2x-4}{9x-7}$
   $\Rightarrow f^{-1}(x) = \frac{2x-4}{9x-7}$
6. $f(x) = \frac{6x}{2-5x}$
   Write this as $y = \frac{6x}{2-5x}$
   Step 1: $x = \frac{6y}{2-5y}$
   Step 2: $x(2-5y) = 6y$
   $\Rightarrow 2x - 5xy = 6y$
   $\Rightarrow 2x = 6y + 5xy$
   $\Rightarrow 2x = y(6+5x)$
   $\Rightarrow \frac{2x}{6+5x} = y$
   $\Rightarrow f^{-1}(x) = \frac{2x}{6+5x}$
7. $f(x) = \frac{1-3x}{3x-8}$
   Write this as $y = \frac{1-3x}{3x-8}$
   Step 1: $x = \frac{1-3y}{3y-8}$
   Step 2: $x(3y-8) = 1 - 3y$
   $\Rightarrow 3xy - 8x = 1 - 3y$
   $\Rightarrow 3xy + 3y = 1 + 8x$
   $\Rightarrow y(3x+3) = 1 + 8x$
   $\Rightarrow y = \frac{1+8x}{3x+3}$
   $\Rightarrow f^{-1}(x) = \frac{1+8x}{3x+3}$

Example

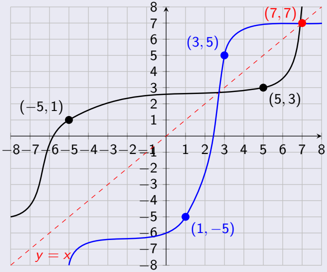
Find the inverse of $y = \sqrt{x}$.
Step 1: $x = \sqrt{y}$
Step 2: $x^2 = y \quad \Rightarrow f^{-1}(x) = x^2$



Points on:

$f(x) = \sqrt{x}$	$f^{-1}(x) = x^2$
(4, 2)	(2, 4)
(9, 3)	(3, 9)

Graph of the inverse function



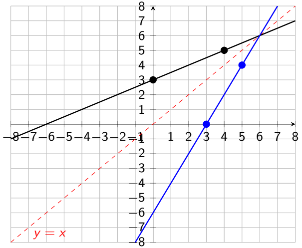
The graph of $y = f^{-1}(x)$ is the reflection of the graph of $y = f(x)$ about the diagonal $y = x$.

Any point (x, y) on the graph of f has a corresponding point (y, x) on the graph on the inverse f^{-1}.

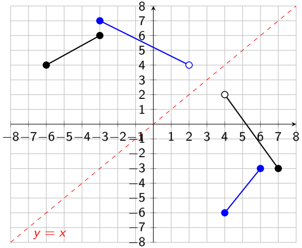
Graphs of inverse functions - exercises

For the given graph of $y = f(x)$, graph the corresponding graph of $y = f^{-1}(x)$.

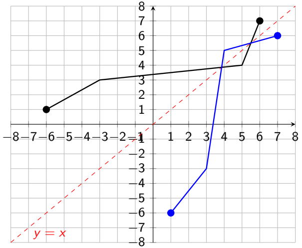
1.



3.



2.



4.