The inverse of a function

Lesson #6

MAT 1375 Precalculus

New York City College of Technology CUNY

Definition

A function $y = f(x)$ is called *one-to-one* if

two different inputs $x_1 \neq x_2$ always give two different outputs $f(x_1) \neq f(x_2)$.

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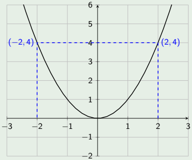
$$f(2) = 4$$
$$f(-2) = 4$$

No! ✗

f is not one-to-one!



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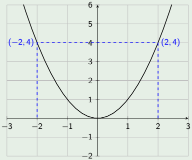
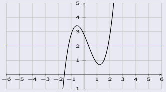


Horizontal line test



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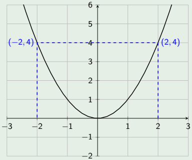

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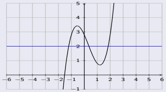

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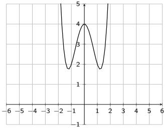
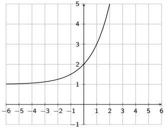
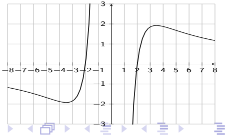
1. 
2. 
3. 

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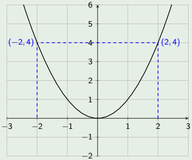

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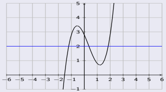

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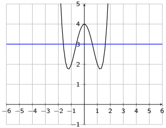
1. 

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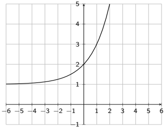
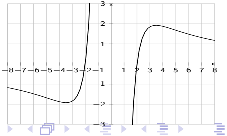
2. 

3. 

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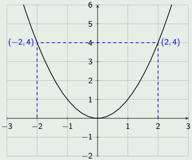

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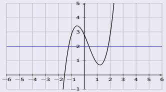

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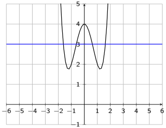
1. 

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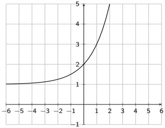
2. 

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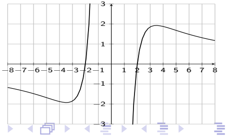
3. 

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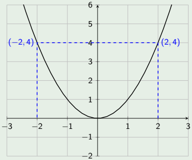

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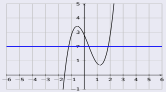

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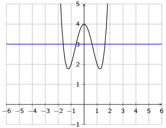
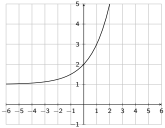
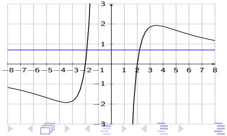
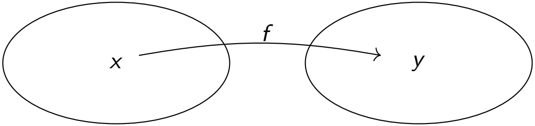
1.  No!
2.  Yes!
3.  No!

The inverse function

If f is one-to-one, then every output y comes from exactly one input x.



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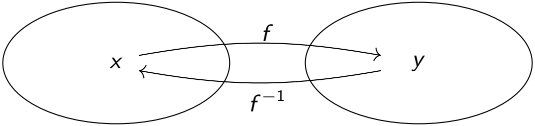

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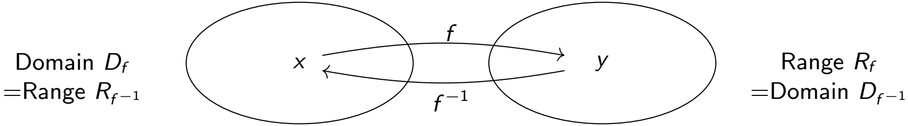

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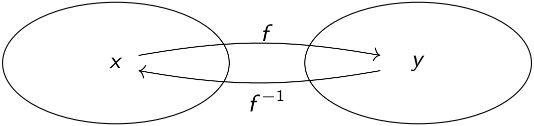
$=$Range $R_{f^{-1}}$



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Definition

Let $y = f(x)$ be a one-to-one function.
Define the *inverse function of f* to be the function f^{-1} which maps

$$y = f(x) \quad \Leftrightarrow \quad x = f^{-1}(y)$$

f^{-1} has domain $D_{f^{-1}} = R_f$, and has range $R_{f^{-1}} = D_f$.

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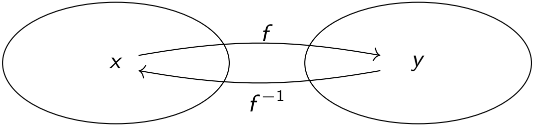

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Note: Composing f and f^{-1} gives back the original input:

$$f^{-1}(f(x)) = x \quad \text{and} \quad f(f^{-1}(x)) = x$$

Even better:
A function g is the inverse of f, that is $g = f^{-1}$, precisely when $D_g = R_f$, $R_g = D_f$, and

$$g(f(x)) = x \quad \text{and} \quad f(g(x)) = x$$

Steps for finding the inverse function

Step 1: Write $y = f(x)$ and switch the xs and ys.
Step 2: Solve for y.

Find the inverse function.

1. $f(x) = 3x - 7$
2. $f(x) = 6x + 4$
3. $f(x) = \sqrt{2x - 5}$

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   $\Rightarrow \frac{x+7}{3} = y$
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   $y = \sqrt{2x - 5}$
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   $\Rightarrow x^2 + 5 = 2y$
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Inverse function - exercises

Find the inverse function.

4. $f(x) = \frac{2x+3}{4x+5}$
6. $f(x) = \frac{6x}{2-5x}$
5. $f(x) = \frac{7x-4}{9x-2}$
7. $f(x) = \frac{1-3x}{3x-8}$

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   Step 1: $x = \frac{2y+3}{4y+5}$

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   $\Rightarrow 4xy + 5x = 2y + 3$

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   Step 1: $x = \frac{1-3y}{3y-8}$
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   $\Rightarrow y = \frac{1+8x}{3x+3}$
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Find the inverse of $y = \sqrt{x}$.

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Step 1: $x = \sqrt{y}$
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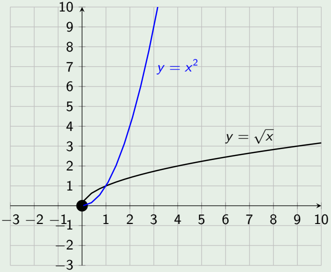

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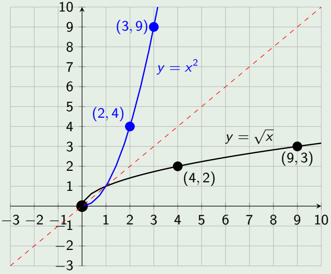

Points on:

$f(x) = \sqrt{x}$	$f^{-1}(x) = x^2$
$(4, 2)$	$(2, 4)$
$(9, 3)$	$(3, 9)$

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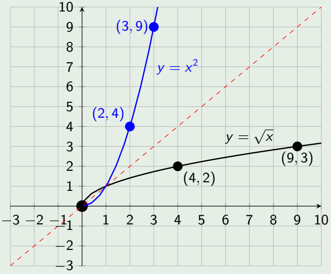
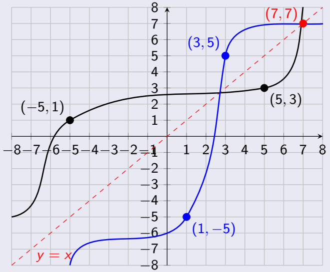


Points on:

$f(x) = \sqrt{x}$	$f^{-1}(x) = x^2$
$(4, 2)$	$(2, 4)$
$(9, 3)$	$(3, 9)$

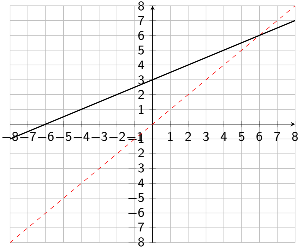
Graph of the inverse function

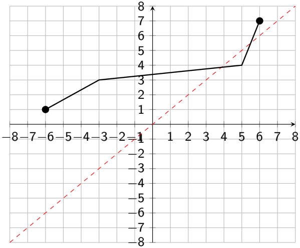


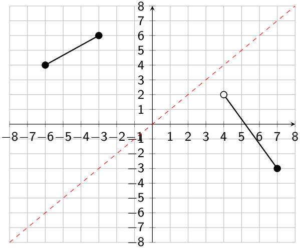
The graph of $y = f^{-1}(x)$ is the reflection of the graph of $y = f(x)$ about the diagonal $y = x$.

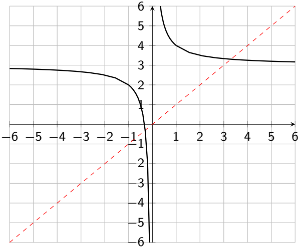
Any point (x, y) on the graph of f has a corresponding point (y, x) on the graph on the inverse f^{-1}.

For the given graph of $y = f(x)$, graph the corresponding graph of $y = f^{-1}(x)$.

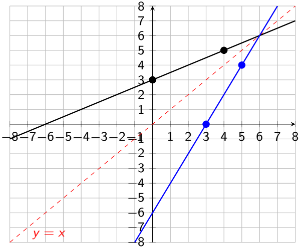
1. 

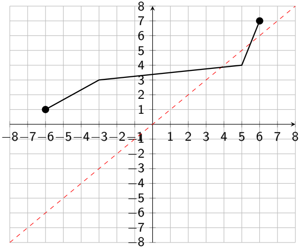
2. 

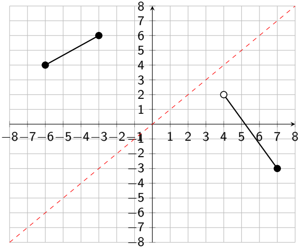
3. 

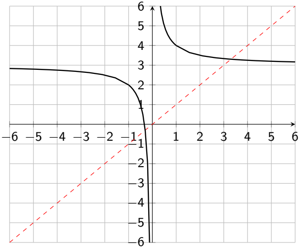
4. 

For the given graph of $y = f(x)$, graph the corresponding graph of $y = f^{-1}(x)$.

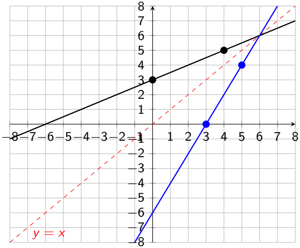
1. 

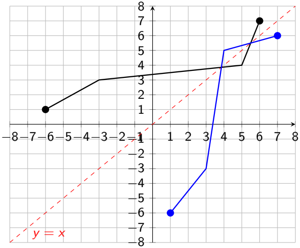
2. 

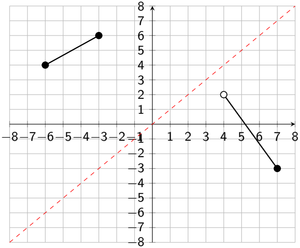
3. 

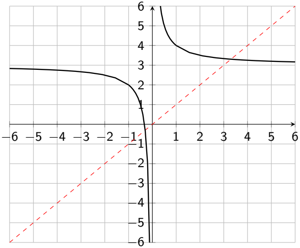
4. 

For the given graph of $y = f(x)$, graph the corresponding graph of $y = f^{-1}(x)$.

1. 

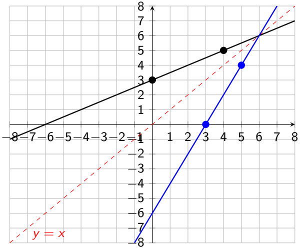
2. 

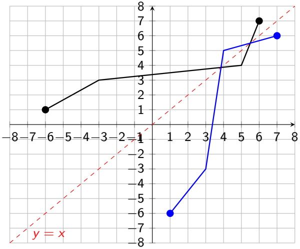
3. 

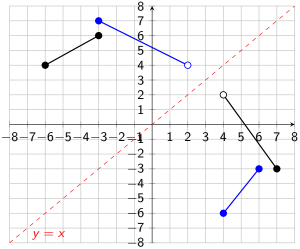
4. 

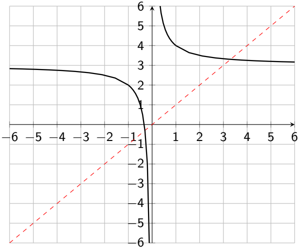
Graphs of inverse functions - exercises

For the given graph of $y = f(x)$, graph the corresponding graph of $y = f^{-1}(x)$.

1. 

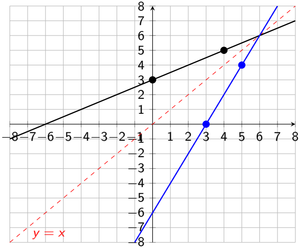
2. 

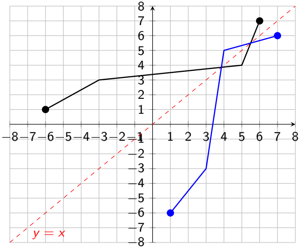
3. 

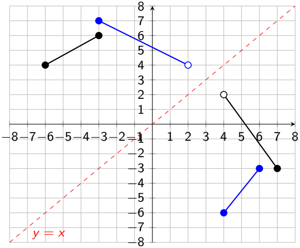
4. 

Graphs of inverse functions - exercises

For the given graph of $y = f(x)$, graph the corresponding graph of $y = f^{-1}(x)$.

1. 

2. 

3. 

4.