Symmetries and operations on functions

Lesson #5

MAT 1375 Precalculus

New York City College of Technology CUNY

Even and odd functions

Even functions

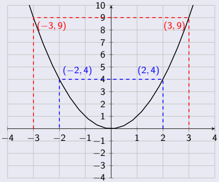
A function is *even*, if

$$f(-x) = f(x) \quad \text{for all } x$$

Example: $f(x) = x^2$

Find $f(2) = 4$ $\quad f(-2) = 4$

$f(3) = 9$ $\quad f(-3) = 9$



Even functions have graphs that are symmetric with respect to the y-axis.

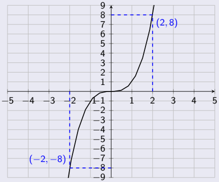
Odd functions

A function is *odd*, if

$$f(-x) = -f(x) \quad \text{for all } x$$

Example: $f(x) = x^3$

Find $f(2) = 8$ $\quad f(-2) = -8$



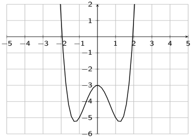
Odd functions have graphs that are symmetric with respect to the origin.

Note: If 0 is in the domain, then $f(0) = f(-0) = -f(0)$, so that:

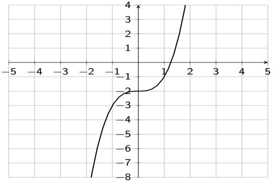
$$f(0) = 0$$

Determine if the function is even, odd, or neither.

1. $f(x) = x^4 - 3x^2 - 3$

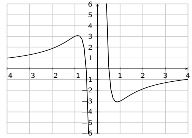


Even

2. 

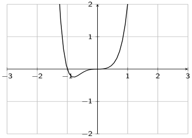
Neither

3. $f(x) = \frac{1}{x^3} - \frac{4}{x}$



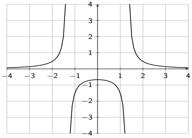
Odd

4. $f(x) = x^3 + x^6$

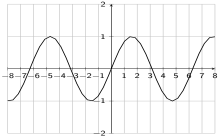


Neither

5. $f(x) = \frac{x^2+2}{x^4-3}$



Even

6. 

Odd

1. Let $f(x) = x^2 + 7x + 12$ and $g(x) = 2x - 6$.
   - Find the sum $(f+g)(x) = f(x) + g(x)$
     $= (x^2+7x+12) + (2x-6) = x^2 + 9x + 6$
   - Find the difference $(f-g)(x) = f(x) - g(x)$
     $= (x^2+7x+12) - (2x-6) = x^2 + 5x + 18$
   - Find the product $(f \cdot g)(x) = f(x) \cdot g(x)$
     $= (x^2 + 7x + 12) \cdot (2x - 6)$
     $= 2x^3 - 6x^2 + 14x^2 - 42x + 24x - 72$
     $= 2x^3 + 8x^2 - 18x - 72$
   - Find the quotient $\frac{f}{g}(x) = \frac{f(x)}{g(x)} = \frac{x^2+7x+12}{2x-6}$
   - Find the quotient $\frac{g}{f}(x) = \frac{g(x)}{f(x)} = \frac{2x-6}{x^2+7x+12}$

   Domains:
   $D_f = \mathbb{R}$
   $D_g = \mathbb{R}$

Domains:
$D_{f+g} = \mathbb{R}$

$D_{f-g} = \mathbb{R}$

$D_{f \cdot g} = \mathbb{R}$

$2x - 6 = 0 \Rightarrow 2x = 6 \Rightarrow x = 3$
$D_{\frac{f}{g}} = \mathbb{R} - \{3\}$

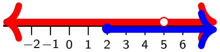
$x^2 + 7x + 12 = 0$
$\Rightarrow (x+3)(x+4) = 0$
$\Rightarrow x = -3$ or $x = -4$
$D_{\frac{g}{f}} = \mathbb{R} - \{-3, -4\}$

2. Let $f(x) = \frac{1}{x-5}$ and $g(x) = \sqrt{x-2}$.
   - Find the sum $(f+g)(x) = f(x) + g(x)$ $= \frac{1}{x-5} + \sqrt{x-2}$
   - Find the difference $(f-g)(x) = f(x) - g(x)$ $= \frac{1}{x-5} - \sqrt{x-2}$
   - Find the product $(f \cdot g)(x) = f(x) \cdot g(x)$ $= \frac{1}{x-5} \cdot \sqrt{x-2} = \frac{\sqrt{x-2}}{x-5}$
   - Find the quotient $\frac{f}{g}(x) = \frac{f(x)}{g(x)} = \frac{\frac{1}{x-5}}{\sqrt{x-2}} = \frac{\frac{1}{x-5}}{\frac{\sqrt{x-2}}{1}}$ $= \frac{1}{x-5} \cdot \frac{1}{\sqrt{x-2}} = \frac{1}{(x-5)\cdot\sqrt{x-2}}$
   - Find the quotient $\frac{g}{f}(x) = \frac{g(x)}{f(x)} = \frac{\sqrt{x-2}}{\frac{1}{x-5}}$ $= \sqrt{x-2} \cdot \frac{x-5}{1} = \sqrt{x-2} \cdot (x-5)$

Domains:

$D_f = \mathbb{R} - \{5\}$

$D_g = [2, \infty)$



Domains:

$D_{f+g} = [2,5) \cup (5,\infty)$

$D_{f-g} = [2,5) \cup (5,\infty)$

$D_{f \cdot g} = [2,5) \cup (5,\infty)$

We need $x \neq 5$, $x - 2 \geq 0$, and $x - 2 \neq 0$

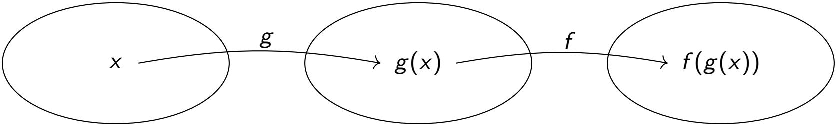
$D_{\frac{f}{g}} = (2,5) \cup (5,\infty)$

We need $x \neq 5$, $x - 2 \geq 0$,

$D_{\frac{g}{f}} = [2,5) \cup (5,\infty)$

Composition

Use the output of the function g as an input for the function f:



Definition

The *composition* $f \circ g$ is the function $(f \circ g)(x) = f(g(x))$.
The domain of $f \circ g$ are all x for which $g(x)$ is defined and then $f(g(x))$ is also defined.

1. $f(x) = 2x + 1$
   $g(x) = x^2 - 5$
   Evaluate:
   $(f \circ g)(4) = f(g(4)) = f(4^2 - 5)$
   $= f(11) = 2 \cdot 11 + 1 = 23$

   $(f \circ g)(3) = f(g(3)) = f(3^2 - 5)$
   $= f(4) = 2 \cdot 4 + 1 = 9$

2. $f(x) = \frac{x+1}{x+2}$
   $g(x) = x + 3$
   Evaluate:
   $(f \circ g)(5) = f(g(5)) = f(5 + 3)$
   $= f(8) = \frac{8+1}{8+2} = \frac{9}{10}$

   $(g \circ f)(5) = g(f(5)) = g(\frac{5+1}{5+2})$
   $= g(\frac{6}{7}) = \frac{6}{7} + 3 = \frac{27}{7}$

3. $f(x) = x^2 - 5x + 8$
   $g(x) = x + 3$
   Evaluate:
   $(f \circ g)(x) = f(g(x)) =$
   $= f(x + 3)$
   $= (x + 3)^2 - 5(x + 3) + 8$
   $= x^2 + 6x + 9 - 5x - 15 + 8$
   $= x^2 + x + 2$

Composition - exercises

Find the compositions and state their domains.

4. $f(x) = x^2 + 3x$
   $g(x) = x - 7$
   $(f \circ g)(x) = f(g(x))$
   $= f(x-7)$
   $= (x-7)^2 + 3 \cdot (x-7)$
   $= x^2 - 14x + 49 + 3x - 21$
   $= x^2 - 11x + 28$
   Domain: $D = \mathbb{R}$

5. $f(x) = \frac{x+2}{x-5}$
   $g(x) = x^2 + 4$
   $(f \circ g)(x) = f(g(x))$
   $= f(x^2 + 4)$
   $= \frac{x^2+4+2}{x^2+4-5} = \frac{x^2+6}{x^2-1}$
   Domain:
   Where is the denominator zero?
   $x^2 - 1 = 0$
   $(x+1)(x-1) = 0 \Rightarrow x = \pm 1$
   $\Rightarrow D = \mathbb{R} - \{-1, +1\}$

6. $f(x) = x^2 + 4x + 6$
   $g(x) = 2x + 3$
   $(f \circ g)(x) = f(g(x))$
   $= f(2x+3)$
   $= (2x+3)^2 + 4 \cdot (2x+3) + 6$
   $= 4x^2 + 12x + 9 + 8x + 12 + 6$
   $= 4x^2 + 20x + 27$
   Domain: $D = \mathbb{R}$

7. $f(x) = x^2 + 4x + 6$
   $g(x) = 2x + 3$
   $(g \circ f)(x) = g(f(x))$
   $= g(x^2 + 4x + 6)$
   $= 2 \cdot (x^2 + 4x + 6) + 3$
   $= 2x^2 + 8x + 12 + 3$
   $= 2x^2 + 8x + 15$
   Domain: $D = \mathbb{R}$

Find the compositions and state their domains.

8. $f(x) = \sqrt{3-2x}$
   $g(x) = 5 - 4x$
   $(f \circ g)(x) = f(g(x))$
   $= f(5-4x)$
   $= \sqrt{3 - 2\cdot(5-4x)}$
   $= \sqrt{3 - 10 + 8x}$
   $= \sqrt{8x-7}$
   Domain:
   $8x - 7 \geq 0 \quad \Rightarrow 8x \geq 7 \quad \Rightarrow x \geq \frac{7}{8}$
   $D = [\frac{7}{8}, \infty)$

9. $f(x) = \sqrt{3-2x}$
   $g(x) = 5 - 4x$
   $(g \circ f)(x) = g(f(x))$
   $= g(\sqrt{3-2x})$
   $= 5 - 4\cdot\sqrt{3-2x}$
   Domain:
   $3 - 2x \geq 0 \quad \Rightarrow -2x \geq -3 \quad \Rightarrow x \leq \frac{3}{2}$
   $D = (-\infty, \frac{3}{2}]$

10. $f(x) = \frac{1}{x^2-x-12}$
    $g(x) = 3x - 5$
    $(g \circ f)(x) = g(f(x))$
    $= g(\frac{1}{x^2-x-12}) = 3 \cdot \frac{1}{x^2-x-12} - 5$
    $= \frac{3}{x^2-x-12} - \frac{5(x^2-x-12)}{x^2-x-12}$
    $= \frac{3-5x^2+5x+60}{x^2-x-12} = \frac{-5x^2+5x+63}{x^2-x-12}$
    Domain:
    $x^2 - x - 12 = 0$
    $\Rightarrow (x+3)(x-4) = 0 \quad \Rightarrow x = -3, x = 4$
    $D = \mathbb{R} - \{-3, 4\}$

11. $f(x) = \frac{1}{x^2-x-12} = \frac{1}{(x+3)(x-4)}$
    $g(x) = 3x - 5$
    $(f \circ g)(x) = f(g(x)) = f(3x-5)$
    $= \frac{1}{((3x-5)+3)\cdot((3x-5)-4)} = \frac{1}{(3x-2)(3x-9)}$
    Domain:
    $3x - 2 = 0$ or $3x - 9 = 0$
    $\Rightarrow x = \frac{2}{3}$ or $x = 3 \quad \Rightarrow D = \mathbb{R} - \{\frac{2}{3}, 3\}$