Functions via formulas

Lesson #2

MAT 1375 Precalculus

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1. Let $f(x) = \frac{x+3}{x+5}$. Find $f(7)$.
   $f(7) = \frac{7+3}{7+5} = \frac{10}{12} = \frac{5}{6}$
2. Let $f(x) = \sqrt{x^2 + 2x}$. Find $f(4)$
   $f(4) = \sqrt{4^2 + 2 \cdot 4} = \sqrt{16 + 8}$
   $= \sqrt{24} = 2\sqrt{6}$
3. Let $f(x) = x^2 + 3x$. Find $f(5)$
   $f(5) = 5^2 + 3 \cdot 5 = 25 + 15 = 40$
4. Let $f(x) = x^2 + 3x$. Find $f(a)$
   $f(a) = a^2 + 3a$
5. Let $f(x) = \begin{cases} 3x + 2 & \text{for } 1 < x \leq 3 \\ x^2 - x & \text{for } 5 \leq x \leq 9 \end{cases}$
   f is a *piecewise defined* function!
   Find $f(2) = 3 \cdot 2 + 2 = 6 + 2 = 8$
   Find $f(3) = 3 \cdot 3 + 2 = 9 + 2 = 11$
   Find $f(4) =$ undefined
   Find $f(5) = 5^2 - 5 = 25 - 5 = 20$
   Find $f(7) = 7^2 - 7 = 49 - 7 = 42$
   Find $f(1) =$ undefined
7. $f(x) = \begin{cases} |x| & \text{for } -6 \leq x < 2 \\ \sqrt{x-1} & \text{for } 2 < x \leq 7 \\ x + 2 & \text{for } 7 < x < 8 \end{cases}$
   Find $f(3) = \sqrt{3-1} = \sqrt{2}$
   Find $f(-\pi) = |-\pi| = \pi$
   Find $f(2) =$ undefined
   Find $f(7) = \sqrt{7-1} = \sqrt{6}$
   Find $f(\sqrt{55}) = \sqrt{55} + 2$
8. Let $f(x) = x^2 + 3x$. Find $f(a+5)$
   $f(a+5) = (a+5)^2 + 3(a+5)$
   $= a^2 + 2 \cdot a \cdot 5 + 25 + 3a + 15$
   $= a^2 + 10a + 25 + 3a + 15$
   $= a^2 + 13a + 40$
9. Let $f(x) = x^2 - x + 7$. Find $f(b-2)$
   $f(b-2) = (b-2)^2 - (b-2) + 7$
   $= b^2 - 4b + 4 - b + 2 + 7$
   $= b^2 - 5b + 13$
10. Let $f(x) = \sqrt{x^2 - 3}$. Find $f(x+h)$.
    $f(x+h) = \sqrt{(x+h)^2 - 3}$
    $= \sqrt{x^2 + 2xh + h^2 - 3}$

Difference quotient

Definition

For a function $y = f(x)$, the *difference quotient* is defined as

$$\frac{f(x+h)-f(x)}{h}$$

- Step 1: Find $f(x+h)$
- Step 2: Find $f(x+h)-f(x)$
- Step 3: Find $\frac{f(x+h)-f(x)}{h}$

1. Let $f(x) = x^2 - 5x$.
   - Step 1: $f(x+h) = (x+h)^2 - 5(x+h) = x^2 + 2xh + h^2 - 5x - 5h$
   - Step 2: $f(x+h) - f(x) = (x^2 + 2xh + h^2 - 5x - 5h) - (x^2 - 5x)$
     $= x^2 + 2xh + h^2 - 5x - 5h - x^2 + 5x = 2xh + h^2 - 5h$
   - Step 3: $\frac{f(x+h)-f(x)}{h} = \frac{2xh+h^2-5h}{h} = \frac{h\cdot(2x+h-5)}{h} = 2x + h - 5$
2. Find the difference quotient of $f(x) = x^2 + 8x$

Find the difference quotients of the following functions.

2. $f(x) = x^2 + 8x$

$$f(x+h) = (x+h)^2 + 8(x+h)$$
$$= x^2 + 2xh + h^2 + 8x + 8h$$
$$f(x+h) - f(x) =$$
$$= (x^2 + 2xh + h^2 + 8x + 8h) - (x^2 + 8x)$$
$$= x^2 + 2xh + h^2 + 8x + 8h - x^2 - 8x$$
$$= 2xh + h^2 + 8h$$
$$\frac{f(x+h)-f(x)}{h} = \frac{2xh+h^2+8h}{h} = \frac{h\cdot(2x+h+8)}{h}$$
$$= 2x + h + 8$$

3. $f(x) = x^2 - 3x + 7$

$$f(x+h) = (x+h)^2 - 3(x+h) + 7$$
$$= x^2 + 2xh + h^2 - 3x - 3h + 7$$
$$f(x+h) - f(x) =$$
$$= (x^2 + 2xh + h^2 - 3x - 3h + 7) - (x^2 - 3x + 7)$$
$$= 2xh + h^2 - 3h$$
$$\frac{f(x+h)-f(x)}{h} = \frac{2xh+h^2-3h}{h} = \frac{h\cdot(2x+h-3)}{h}$$
$$= 2x + h - 3$$

4. $f(x) = 3x^2 - 2x - 5$

$$f(x+h) = 3(x+h)^2 - 2(x+h) - 5$$
$$= 3(x^2 + 2xh + h^2) - 2x - 2h - 5$$
$$= 3x^2 + 6xh + 3h^2 - 2x - 2h - 5$$
$$f(x+h) - f(x) = 6xh + 3h^2 - 2h$$
$$\frac{f(x+h)-f(x)}{h} = \frac{6xh+3h^2-2h}{h}$$
$$= \frac{h\cdot(6x+3h-2)}{h} = 6x + 3h - 2$$

5. $f(x) = -5x^2 - x + 9$

$$f(x+h) = -5(x+h)^2 - (x+h) + 9$$
$$= -5(x^2 + 2xh + h^2) - x - h + 9$$
$$= -5x^2 - 10xh - 5h^2 - x - h + 9$$
$$f(x+h) - f(x) = -10xh - 5h^2 - h$$
$$\frac{f(x+h)-f(x)}{h} = \frac{-10xh-5h^2-h}{h}$$
$$= \frac{h\cdot(-10x-5h-1)}{h} = -10x - 5h - 1$$

Difference quotient (polynomial, rational, square root functions) - exercises

Find the difference quotients of the following functions.

6. $f(x) = x^3$

$$= (x+h)(x+h)(x+h)$$
$$= (x^2+2xh+h^2)(x+h)$$
$$= x^3 + x^2h + 2x^2h + 2xh^2 + xh^2 + h^3$$
$$= x^3 + 3x^2h + 3xh^2 + h^3$$
$$f(x+h) - f(x) = 3x^2h + 3xh^2 + h^3$$
$$\frac{f(x+h)-f(x)}{h} = \frac{3x^2h+3xh^2+h^3}{h}$$
$$= \frac{h\cdot(3x^2+3xh+h^2)}{h} = 3x^2 + 3xh + h^2$$

7. $f(x) = \frac{2}{3x-4}$

$$f(x+h) = \frac{2}{3(x+h)-4} = \frac{2}{3x+3h-4}$$
$$f(x+h) - f(x)$$
$$= \frac{2}{3x+3h-4} - \frac{2}{3x-4}$$
$$= \frac{2(3x-4)-2(3x+3h-4)}{(3x+3h+4)(3x-4)}$$
$$= \frac{6x-8-6x-6h+8}{(3x+3h+4)(3x-4)}$$
$$= \frac{-6h}{(3x+3h+4)(3x-4)}$$
$$\frac{f(x+h)-f(x)}{h} = \frac{-6h}{h\cdot(3x+3h+4)(3x-4)}$$
$$= \frac{-6}{(3x+3h+4)(3x-4)}$$

8. $f(x) = \frac{1}{7x+2}$

$$f(x+h) = \frac{1}{7(x+h)+2} = \frac{1}{7x+7h+2}$$
$$f(x+h) - f(x)$$
$$= \frac{1}{7x+7h+2} - \frac{1}{7x+2}$$
$$= \frac{(7x+2)-(7x+7h+2)}{(7x+7h+2)(7x+2)}$$
$$= \frac{-7h}{(7x+7h+2)(7x+2)}$$
$$\frac{f(x+h)-f(x)}{h} = \frac{-7}{(7x+7h+2)(7x+2)}$$

9. $f(x) = \sqrt{2x+3}$

$$f(x+h) = \sqrt{2(x+h)+3} = \sqrt{2x+2h+3}$$
$$f(x+h) - f(x) = \sqrt{2x+2h+3} - \sqrt{2x+3}$$
$$\frac{f(x+h)-f(x)}{h} = \frac{\sqrt{2x+2h+3}-\sqrt{2x+3}}{h}$$
$$= \frac{(\sqrt{2x+2h+3}-\sqrt{2x+3})}{h} \cdot \frac{(\sqrt{2x+2h+3}+\sqrt{2x+3})}{(\sqrt{2x+2h+3}+\sqrt{2x+3})}$$
$$= \frac{\sqrt{2x+2h+3}^2-\sqrt{2x+3}^2}{h\cdot(\sqrt{2x+2h+3}+\sqrt{2x+3})}$$
$$= \frac{2x+2h+3-2x-3}{h\cdot(\sqrt{2x+2h+3}+\sqrt{2x+3})}$$
$$= \frac{2h}{h\cdot(\sqrt{2x+2h+3}+\sqrt{2x+3})}$$
$$= \frac{2}{\sqrt{2x+2h+3}+\sqrt{2x+3}}$$

Standard domain

- $f(x) = \frac{2}{x-3}$. Find $f(8) = \frac{2}{8-3} = \frac{2}{5}$ ✓
  ✗ $f(3) = \frac{2}{0}$ is undefined!
  $\Rightarrow$ 3 is not an allowed input!
  The domain is $D = (-\infty, 3) \cup (3, \infty)$.

- $f(x) = \sqrt{x-5}$. Find $f(9) = \sqrt{4} = 2$ ✓
  ✗ $f(1) = \sqrt{-4}$ is undefined!
  $\Rightarrow$ 1 is not an allowed input: we don't want complex numbers as outputs!
  Domain: $x - 5 \geq 0 \Rightarrow x \geq 5$
  $\Rightarrow D = [5, \infty)$

Standard domain

If no domain is specified, then we assume it to be the *standard domain*, which is the largest possible domain for which our outputs are real numbers.

- fractions $\frac{g(x)}{h(x)}$ must have non-zero denominator $h(x) \neq 0$
- square roots $\sqrt{g(x)}$ must have positive or zero arguments $g(x) \geq 0$
- polynomials are defined for all real numbers

Find the domain of the given function:

1. $f(x) = \frac{x+4}{x+5}$
   When is the denominator $= 0$?
   $x + 5 = 0 \implies x = -5$
   $D = (-\infty, -5) \cup (-5, \infty) = \mathbb{R} - \{-5\}$

2. $f(x) = \frac{7x+3}{6x-4}$
   When is the denominator $= 0$?
   $6x - 4 = 0 \implies 6x = 4$
   $\implies x = \frac{4}{6} = \frac{2}{3}$
   $D = (-\infty, \frac{2}{3}) \cup (\frac{2}{3}, \infty) = \mathbb{R} - \{\frac{2}{3}\}$

3. $f(x) = \frac{2}{x^2+5x-14}$
   When is the denominator $= 0$?
   $x^2 + 5x - 14 = 0$
   $\implies (x-2)(x+7) = 0$
   $\implies x = 2$ or $x = -7$
   $D = (-\infty, -7) \cup (-7, 2) \cup (2, \infty) = \mathbb{R} - \{-7, 2\}$

4. $f(x) = \sqrt{2x+6}$
   $2x + 6 \geq 0 \Rightarrow 2x \geq -6 \Rightarrow x \geq -3$
   $D = [-3, \infty)$

5. $f(x) = 5 \cdot \sqrt{7 - 3x}$
   $7 - 3x \geq 0 \Rightarrow -3x \geq -7$
   $\Rightarrow x \leq \frac{-7}{-3} = \frac{7}{3}$
   $D = (-\infty, \frac{7}{3}]$

6. $f(x) = x^3 + 4x^2 - 2x + 7$
   $D = \mathbb{R}$

7. $f(x) = \begin{cases} 3x & \text{for } 2 < x \leq 5 \\ x^2 & \text{for } 7 \leq x < 9 \\ x + 5 & \text{for } 12 \leq x \end{cases}$
   $D = (2, 5] \cup [7, 9) \cup [12, \infty)$