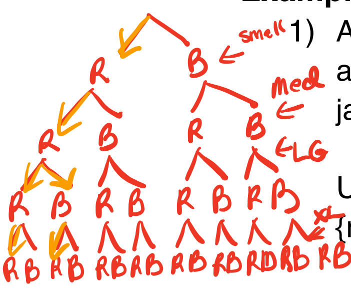
9/23/2021

Multiplication Principle

If n operations $O_1, O_2, \ldots, O_n$ are performed in order, with possible number of outcomes $N_1, N_2, \ldots, N_n$ respectively, then there are $N_1 \cdot N_2 \cdot \cdots \cdot N_n$ possible combined outcomes of the operations performed in the given order.

For a sequence of events!

Examples



1) A retail store stocks windbreaker jackets in small, medium, large, and extra-large, and all are available in blue or red. We choose small, medium, large and xl jackets. How many choices do we have? What are they?

   $\underset{\text{Small}}{\underline{2}} \cdot \underset{\text{Med}}{\underline{2}} \cdot \underset{\text{LG}}{\underline{2}} \cdot \underset{\text{XL}}{\underline{2}}$

   Use the fundamental counting principle to get: 2*2*2*2=16
   {rrrr, bbbb, rrbb, rbbr, rbrb, brbr,…etc.}

2) From the 26 letters in the alphabet, how many ways can 3 letters appear in a row on a license plate if no letter is repeated? If letters can be repeated? If adjacent letters cannot be alike?

   Using the Fundamental Counting Principle

   No letter repeated: $\underset{1^{st}}{\underline{26}} \cdot \underset{2^{nd}}{\underline{25}} \cdot \underset{3^{rd}}{\underline{24}} = 15{,}600$

   Letters can be repeated: $\underset{1^{st}}{\underline{26}} \cdot \underset{2^{nd}}{\underline{26}} \cdot \underset{3^{rd}}{\underline{26}} = 17{,}576$

   Adjacent letters not alike: $\underset{1^{st}}{\underline{26}} \cdot \underset{2^{nd}}{\underline{24}} \cdot \underset{3^{rd}}{\underline{26}} = 16{,}224$

3) How many 5-digit ZIP code numbers are possible if successive digits must be different?

   digits: 0, 1, 2, …, 9 → total of 10 digits

   Again, use the Fundamental Counting Principle

   $\underset{\substack{1^{st}\\ \#\ \text{choices}}}{\underline{10}} \cdot \underset{\substack{2^{nd}\\ \#\ \text{choices}}}{\underline{9}} \cdot \underset{3^{rd}}{\underline{9}} \cdot \underset{4^{th}}{\underline{9}} \cdot \underset{5^{th}}{\underline{9}} = \boxed{65{,}610}$

Factorials

The product of the first n natural numbers is called n factorial and is denoted by $n!$.

For n a natural number,

$n! = n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 2 \cdot 1$

$0! = 1$

$n! = n \cdot (n-1)!$

Permutations

A permutation of a set of distinct objects is an arrangement of the objects in a specific order without repetition.

Example

Suppose that 4 pictures are to be arranged from left to right on one wall of an art gallery. How many permutations (ordered arrangements) are possible?

$$\underset{1^{st}}{\underline{4}} \cdot \underset{2^{nd}}{\underline{3}} \cdot \underset{3^{rd}}{\underline{2}} \cdot \underset{4^{th}}{\underline{1}} = 4! = 24$$

choices

Permutations of n Objects Taken r at a Time

A permutation of a set of n distinct objects taken r at a time without repetition is an arrangement of r of the n objects in a specific order.

$$P_{n,r} = \frac{n!}{(n-r)!} = \frac{n!}{(n-r)!}$$

where $0 \leq r \leq n$.

Note that $P_{n,n} = n!$

Example

Now suppose that the director of the art gallery decides to use only 2 of the 4 available paintings, and they will be arranged on the wall from left to right. How many permutations of 2 paintings can be formed from the 4?

$$_4P_2 = \frac{4!}{(4-2)!} = \frac{4\cdot 3\cdot \cancel{2\cdot 1}}{\cancel{2\cdot 1}} = \boxed{12}$$

Combinations

A combination of a set of n distinct objects taken r at a time without repetition is an r-element subset of the set of n objects. The arrangement of the elements in the subset does NOT matter.

$$C_{n,r} = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

"Binomial coefficient"

where $0 \le r \le n$.

Example

Now suppose that an art museum owns 8 paintings and another museum wishes to borrow 3 of these paintings for a special show. How many ways can the 3 paintings be selected for shipment out of 8 available? When shipping the paintings, is the order important?

Order does not matter, so use combinations. (not permutations)

$$_8C_3 = C(8,3) = \binom{8}{3} = 56$$

Experiments with Equally Likely Outcomes

Each outcome in the sample space S is equally likely to occur. That is, if sample space S consists of n outcomes, say, $S = \{E_1, E_2, E_3, \cdots, E_n\}$, then $P(E_1) = P(E_2) = \cdots = P(E_n)$.

The probability of any event A is equal to the proportion of the outcomes in the sample space that is in A. That is,

$$P(A) = \frac{\text{number of outcomes in } S \text{ that are in } A}{n}$$

Remember: probabilities are values $0 \leq P \leq 1$

Examples:

1) Rolling a die with $S = \{1, 2, 3, 4, 5, 6\}$. Compute the probability of the events $E_1 = \{1, 3, 5\}$ and $E_2 = \{6\}$.

$P(E_1) = \frac{3}{6} = .5$

$P(E_2) = \frac{1}{6}$

2) Flipping two coins: $S = \{HH, HT, TH, TT\}$. Compute the probability of following events.

(a) Getting at least one H. $= \{HH, HT, TH\}$

$P(\text{at least one } H) = \frac{3}{4} = \boxed{.75}$

(b) Getting exactly one H.

$E_2 = \{HT, TH\}$ $P(E_2) = \frac{2}{4} = \boxed{.5}$

(c) Getting at least one H or at least one T.

$E_3 = \{HH, HT, TH, TT\}$ $P(E_3) = \frac{4}{4} = 1$

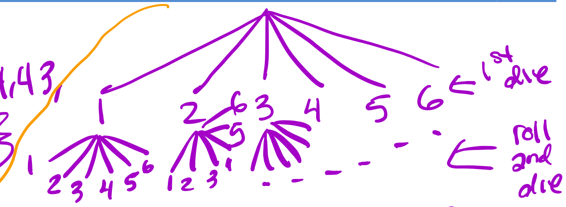
(d) Getting three T. $P(3T\text{'s}) = 0$

3) Consider an experiment of rolling two die and compute the probability of following events.

(a) Producing a sum of 7.

$E_1 = \text{sum of } 7 = \{16, 61, 34, 43, 52, 25\}$

$\boxed{P(E_1) = \frac{6}{36}}$



$\{11, 12, 13, \ldots, 21, 22, 23, \ldots, 31, 32, \ldots\}$

$n = 6 \cdot 6 = 36$

(b) Producing a sum of 11.

$E_2 = \text{sum of } 11 = \{56, 65\}$

$P(E_2) = \frac{2}{36} = \boxed{\frac{1}{18}}$

(c) Producing sum that is NOT 7 or 11.

$E_3 = \text{sum } 7 \text{ or } 11 = \{16, 61, 34, 43, 52, 25, 56, 65\}$

$P(E_3) = \frac{8}{36} = \boxed{\frac{2}{9}}$

In craps this is called a "come out" roll

win if sum = 7 or 11 $P(\text{win}) = \boxed{\frac{2}{9}}$

LOSE if sum = 2, 3 or 12 = E_4

$P(\text{LOSE}) = \frac{4}{36} = \boxed{\frac{1}{9}}$

$E_4 = \{11, 21, 12, 66\}$

Any other roll becomes the "point" and must come up again before 7 is rolled to win!

(d) Producing a sum less than 4.

you do this!

(e) Rolling the same number for both dice.

you do this ("snake eyes" 11)

4) Among 32 dieters following a similar routine, 18 lost weight, 5 gained weight, and 9 remained the same weight. If one of these dieters is randomly chosen, find the probability that he or she gained weight, lost weight, and neither lost nor gained weight.

Exercises (Counting Principles)

1) From a committee of 10 people

(a) In how many ways can we choose a chairperson, a vice-chairperson, and a secretary, assuming that one person cannot hold more than one position?

(b) In how many ways can we choose a subcommittee of 3 people?

2) A group of 100 people touring Europe includes 55 people who speak French, 42 who speak German, and 16 who speak neither language. How many people in the group speak both French and German?

3) From a standard 52-card deck, how many 7-card hands will have all diamonds?

In a standard deck → 13 diamond cards

Here order does NOT matter

so use combinations $C(13,7) = 1716$

4) How many 5-card hands will have 3 aces and 2 kings?

Aces = 4

Kings = 4

$\underline{C(4,3)} \cdot \underline{C(4,2)} = \boxed{24}$

$\rightarrow \dfrac{24}{C(52,5)} = \dfrac{24}{2{,}598{,}960}$

Exercises (Probability)

1) In a family with 3 children, excluding multiple births, what is the probability of having 3 girls? Assume that a girl is as likely as a boy at each birth.

2) Consider an experiment consists of drawing 5 cards from 52-card deck. Compute the probability of following hands:

(a) 5 face cards.

In a standard deck → 12 face cards

$$P = \frac{C(12,5)}{C(52,5)} = \frac{792}{2,598,960} = .000304..$$

(b) 4 aces.

4 aces total → 52−4 = 48 "non aces"

$$\boxed{P(4\text{ aces}) = \frac{C(4,4)\cdot C(48,1)}{C(52,5)}} = \frac{48}{2,598,960}$$

(c) Royal flush (10, J, Q, K, A all in one suit).

There are only 4 suits ⇒ so there are only 4 ways this can happen

$$P(\text{Royal Flush}) = \frac{4}{2,598,960}$$

(d) 2 aces and 3 queens.

don't care about order

4 aces total

4 queens total

$$\frac{C(4,2)\cdot C(4,3)}{C(52,5)}$$

2+3=5

(e) 2 kings and 3 aces.

4 Kings total

4 aces total

$$\frac{C(4,2)\cdot C(4,3)}{C(52,5)}$$

3) From a standard 52-card deck, what is the probability of 5-card hand having at least one face card?

There are 40 non-face cards.

$$P(0) = \frac{C(40,5)}{C(52,5)}$$

P(at least one face card)

$$= P(1)+P(2)+P(3)+P(4)+P(5)$$

$$\text{or} = 1 - P(\text{zero face cards}) = \boxed{1-\frac{C(40,5)}{C(52,5)}}$$

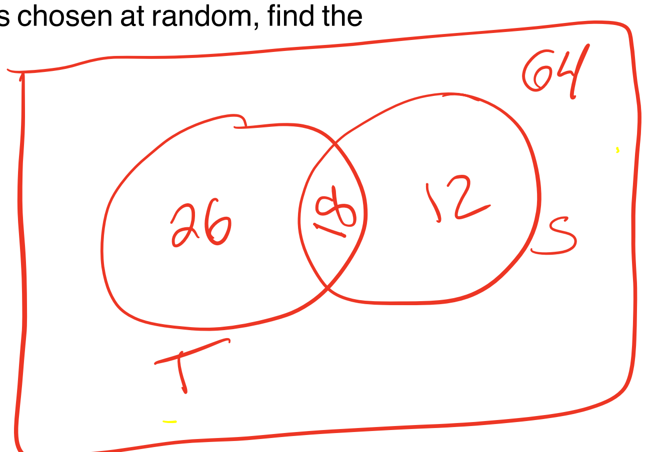
4) A sports club has 120 members, of whom 44 play tennis, 30 play squash, and 18 play both tennis and squash. If a member is chosen at random, find the probability that this person.

(a) Does not play tennis.

Venn diagram

120 Total
44 tennis
30 squash
18 both

120 − (26+18+12) = 64



(b) Does not play squash.

a) $P(\text{no tennis}) = \frac{76}{120}$ b) $P(\text{no squash}) = \frac{90}{120}$

(c) Plays neither tennis nor squash.

$P(\text{no tennis no squash}) = \frac{64}{120}$

Challenging

* 5) A shipment of 55 precision parts, including 12 that are defective, is sent to an assembly plant. The quality control division selects 10 at random for testing and rejects the entire shipment if one or more in the sample are found defective. What is the probability that the shipment will be rejected?