NEW YORK CITY COLLEGE OF TECHNOLOGY
CITY UNIVERSITY OF NEW YORK

An Introduction to Trigonometry

Preparation for
MAT 1275: College Algebra and Trigonometry

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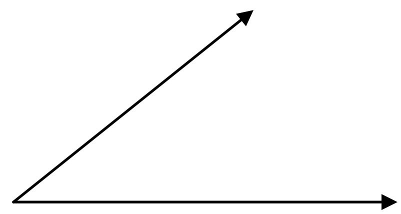
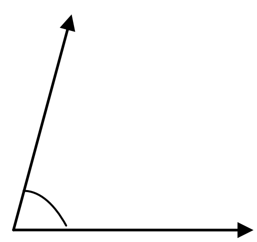
Section 1: Angles

1. An **angle** is the joining of two rays at a common endpoint called the vertex.



2. Angles can be named using a letter at the vertex, a Greek letter, or the letters from the rays forming the sides. Examples: ____ .



3. Types of angles

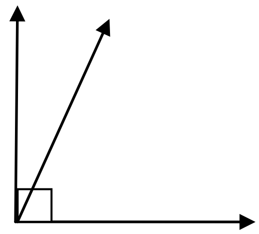
a) Right angle

b) Straight angle

c) Acute angle

d) Obtuse angle

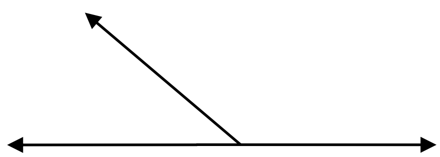
e) **Complementary angles** are two angles that sum to ________________________ .



Example1. Give the complement for each angle:

i) 42°   ii) 83°   iii) 56°

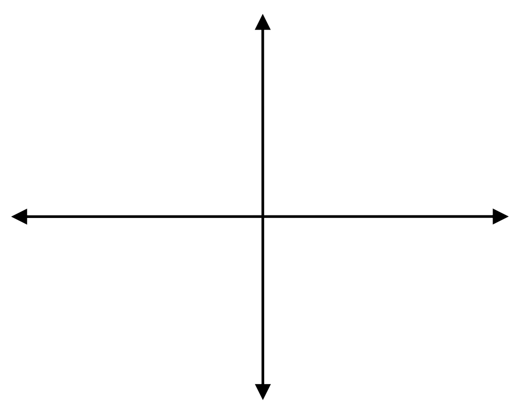
f) **Supplementary angles** are two angles that sum to _________________________ .



Example2. Give the supplement for each angle:

i) 42°   ii) 83°   iii) 118°

4. Quadrants



Quadrant I

x is _____ and y is ______

Quadrant II

x is _____ and y is ______

Quadrant III

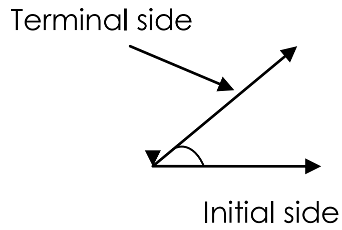
x is _____ and y is ______

Quadrant IV

x is _____ and y is ______

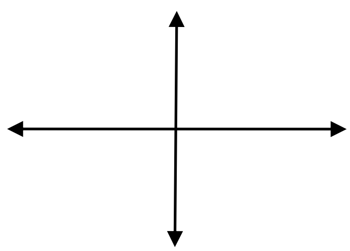
5. A **degree** is _______________ of one complete rotation.

Angles drawn in a counterclockwise direction are ___________________ angles.

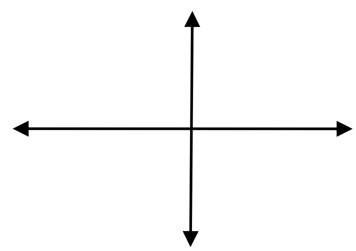


Example3. Draw the following angle:

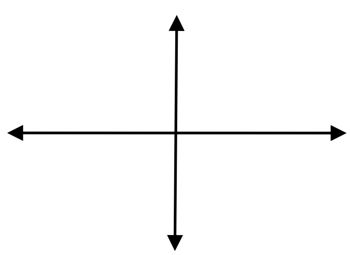
i) $60°$



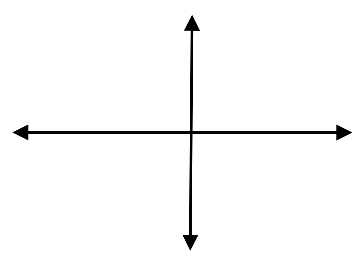
ii) $150°$



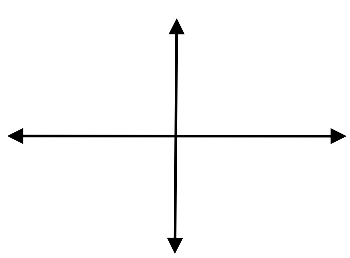
iii) $210°$



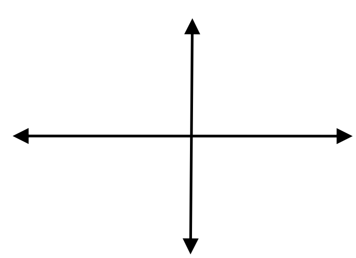
iv) $300°$



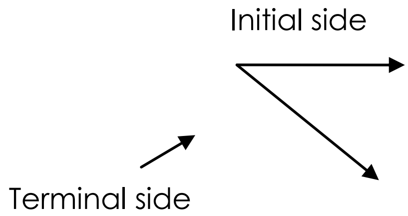
v) $225°$



vi) $270°$

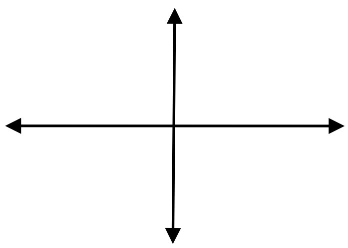


Angles drawn in a clockwise direction are __________________ angles.

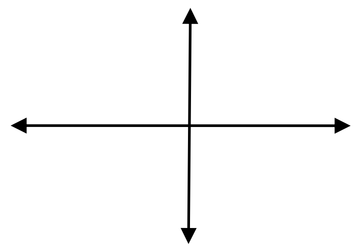


Example 4. Draw the following angle:

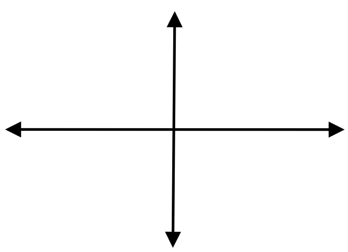
i) $-45°$



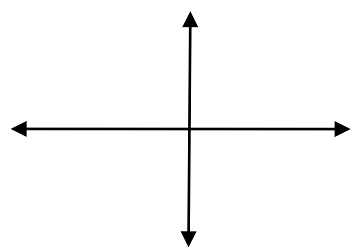
ii) $-120°$



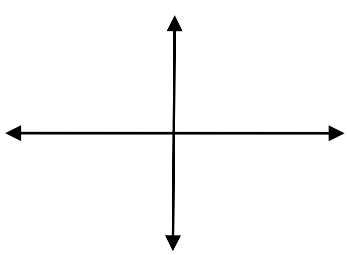
iii) $-225°$



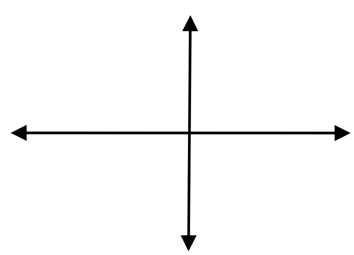
ii) $-330°$



v) $-150°$



vi) $-180°$



5. Coterminal angles are angle that share the same ____________________ and ____________________ sides.

Coterminal angles will always differ by multiples of $360°$.

The formula to find positive coterminal angles is $\theta + 360n$, where n =1, 2, 3, …

The formula to find negative coterminal angles is $\theta - 360n$, where n =1, 2, 3, …

Example 5. Find two positive and two negative coterminal angles for each angle.

i) $45°$

ii) $57°$

iii) $145°$

iv) $300°$

v) $-75°$

vi) $-130°$

vii) $-290°$

SECTION 1 SUPPLEMENTARY EXERCISES:

1. Find the complement for each angle.

a) $73°$

b) $8°$

c) $45°$

d) $30°$

e) $55°$

f) $28°$

2. Find the supplement for each angle.

a) $6°$

b) $99°$

c) $101°$

d) $45°$

e) $115°$

f) $137°$

3. Find two positive and two negative coterminal angles for each angle.

a) $10°$

b) $104°$

c) $195°$

d) $315°$

e) $-122°$

f) $-247°$

g) $-60°$

h) $-75°$

i) $-172°$

j) $-320°$

k) $135°$

Section 2: Radians and Degrees

1. Converting from Degrees to Radians

To change from degrees to radians- multiply by $\frac{\pi}{180}$

Example1. Change the degree measurements to radians.

i) $45°$

ii) $30°$

iii) $60°$

iv) $225°$

v) $310°$

vi) $28°$

vii) $140°$

viii) $-30°$

vii) $-270°$

vii) $-120°$

2. Converting from Radians to Degrees

To change from radians to degrees- multiply by $\frac{180}{\pi}$

Example2. Change the radian measurements to degrees.

i) $\frac{\pi}{4}$

ii) $\frac{3\pi}{4}$

iii) $\frac{2\pi}{3}$

iv) $\frac{11\pi}{6}$

v) $\frac{7\pi}{4}$

vi) $-\frac{4\pi}{3}$

vii) $-\frac{7\pi}{6}$

viii) $-\frac{5\pi}{12}$

SECTION 2 SUPPLEMENTARY EXERCISES:

1. Change the degree measurements to radians.

a) $120°$

b) $270°$

c) $-12°$

d) $-330°$

e) $18°$

f) $280°$

g) 315°

h) -160°

i) -290°

2. Change the radian measurements to degrees.

a) $\frac{7\pi}{4}$

b) $\frac{4\pi}{3}$

c) $\frac{11\pi}{12}$

d) $-\frac{5\pi}{6}$

e) $\frac{13\pi}{18}$

f) $-\pi$

g) 2π

h) $\frac{13\pi}{4}$

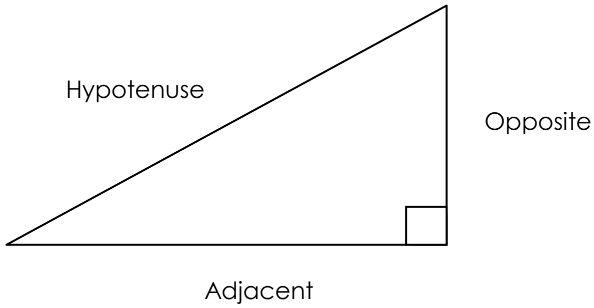
i) $-\frac{5\pi}{18}$

j) $-\frac{11\pi}{6}$

k) $\frac{17\pi}{12}$

Section 3: Defining Trigonometric Functions

1. Trigonometric functions:



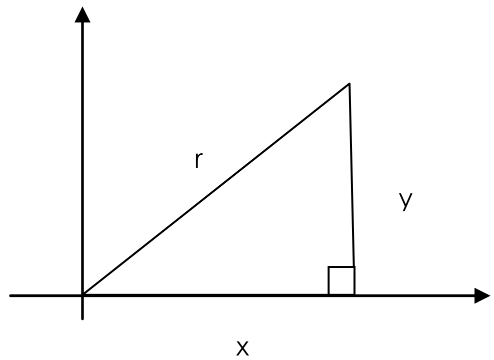
$\sin\theta =$ $\csc\theta =$

$\cos\theta =$ $\sec\theta =$

$\tan\theta =$ $\cot\theta =$

SOH **CAH** **TOA**

2. Trigonometric functions:



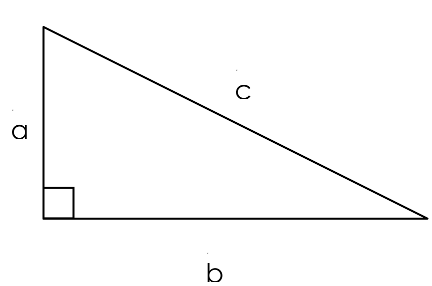
$\sin\theta = \frac{y}{r}$ $\csc\theta = \frac{r}{y}$

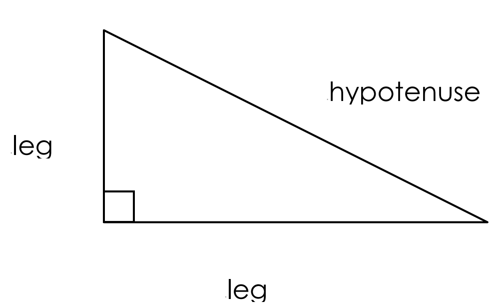
$\cos\theta = \frac{x}{r}$ $\sec\theta = \frac{r}{x}$

$\tan\theta = \frac{y}{x}$ $\cot\theta = \frac{x}{y}$

3. The **Pythagorean Theorem:** For any right triangle with legs a, b and hypotenuse c,

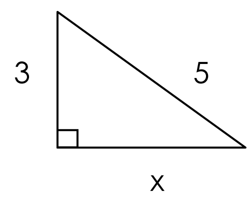
$c^2 = a^2 + b^2$ or $hypotenuse^2 = leg^2 + leg^2$



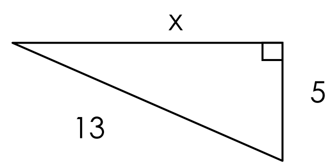


Example1. Find the missing side of the right triangle.

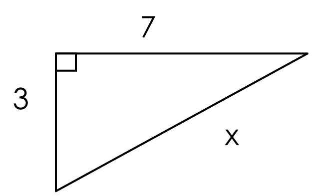
i)



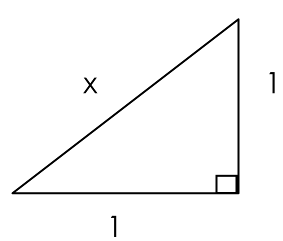
ii)



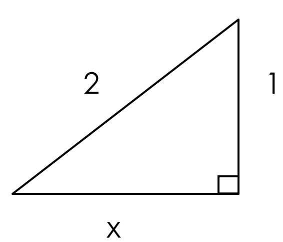
iii)



iv)



v)



4. Reciprocal Identities

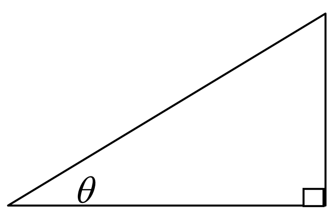
$$\sin\theta = \frac{1}{\csc\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

$$\cos\theta = \frac{1}{\sec\theta} \qquad \sec\theta = \frac{1}{\cos\theta}$$

$$\tan\theta = \frac{1}{\cot\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

Example 2. Assume all the triangles are right triangles.

i) Given $\sin\theta = \frac{1}{2}$ find the other five trigonometric functions.

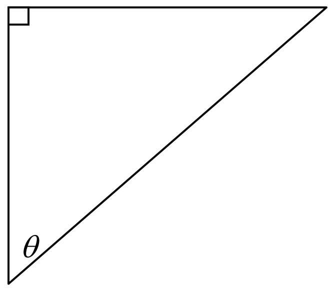


$\sin\theta =$ $\qquad$ $\csc\theta =$

$\cos\theta =$ $\qquad$ $\sec\theta =$

$\tan\theta =$ $\qquad$ $\cot\theta =$

ii) Given $\tan\theta = 1$ find the other five trigonometric functions.

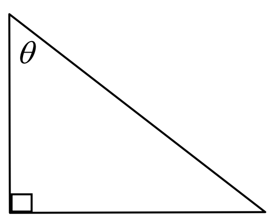


$\sin\theta =$ $\csc\theta =$

$\cos\theta =$ $\sec\theta =$

$\tan\theta =$ $\cot\theta =$

iii) Given $\cos\theta = \frac{\sqrt{3}}{2}$ find the other five trigonometric functions.

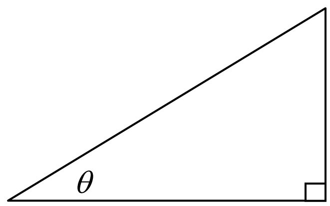


$\sin\theta =$ $\csc\theta =$

$\cos\theta =$ $\sec\theta =$

$\tan\theta =$ $\cot\theta =$

iv) Given $\sin\theta = \frac{\sqrt{7}}{4}$ find the other five trigonometric functions.

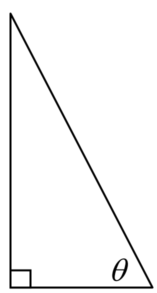


$\sin\theta =$ $\csc\theta =$

$\cos\theta =$ $\sec\theta =$

$\tan\theta =$ $\cot\theta =$

v) Given $\tan\theta = \frac{5}{2}$ find the other five trigonometric functions.

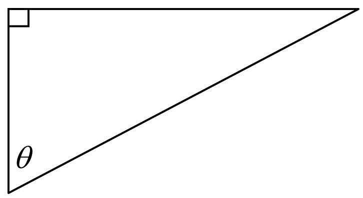


$\sin\theta =$ $\csc\theta =$

$\cos\theta =$ $\sec\theta =$

$\tan\theta =$ $\cot\theta =$

vi) Given $\cos\theta = \frac{\sqrt{5}}{5}$ find the other five trigonometric functions.

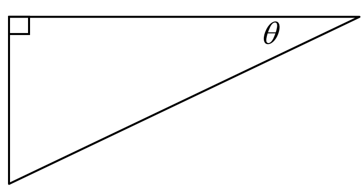


$\sin\theta =$ $\csc\theta =$

$\cos\theta =$ $\sec\theta =$

$\tan\theta =$ $\cot\theta =$

vi) Given $\sec\theta = \frac{4}{3}$ find the other five trigonometric functions.



$\sin\theta =$ $\csc\theta =$

$\cos\theta =$ $\sec\theta =$

$\tan\theta =$ $\cot\theta =$

SECTION 3 SUPPLEMENTARY EXERCISES:

1. In a right triangle, if $\sin\theta = \frac{12}{13}$ find the other five trigonometric functions.

2. In a right triangle, if $\cos\theta = \frac{7}{24}$ find the other five trigonometric functions.

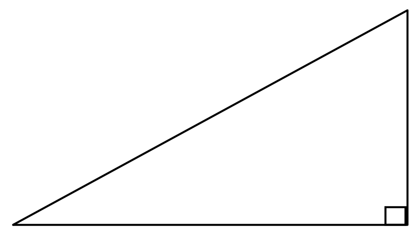
3. In a right triangle, if $\tan\theta = \frac{3}{2}$ find the other five trigonometric functions.

4. In a right triangle, if $\sin\theta = \frac{\sqrt{5}}{7}$ find the other five trigonometric functions.

5. In a right triangle, if $\tan\theta = \frac{\sqrt{3}}{5}$ find the other five trigonometric functions.

Section 4: Trigonometric Functions of Special Angles

1. $30°, 45°, 60°$ **Trigonometric Functions**



$\sin 30° =$

$\cos 30° =$

$\tan 30° =$

$\csc 30° =$

$\sec 30° =$

$\cot 30° =$

$\sin 60° =$

$\cos 60° =$

$\tan 60° =$

$\csc 60° =$

$\sec 60° =$

$\cot 60° =$

$\sin 45° =$

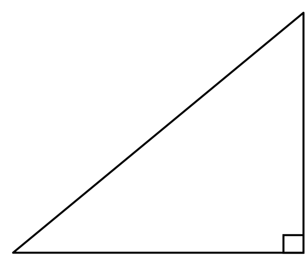
$\cos 45° =$

$\tan 45° =$

$\csc 45° =$

$\sec 45° =$

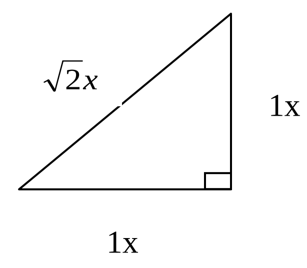
$\cot 45° =$



2. $45^\circ - 45^\circ - 90^\circ$ **Triangles**

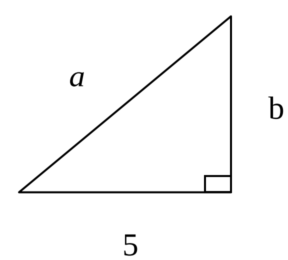
Given a $45^\circ - 45^\circ - 90^\circ$ triangle with one side of length x, the relationship between the corresponding sides is:

$$1x : 1x : \sqrt{2}x$$

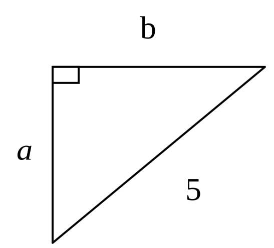


Example1. Find the missing sides.

i)



ii)



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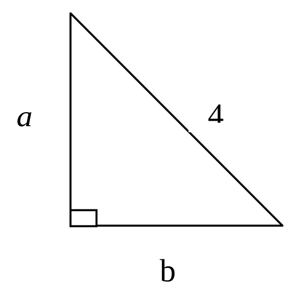
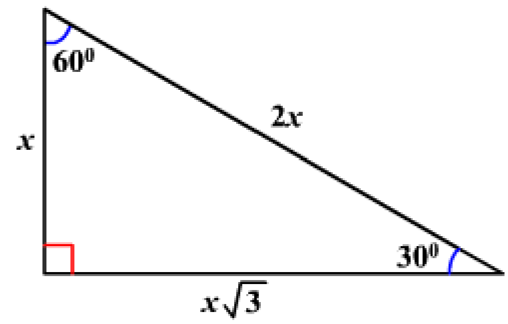
iii)



3. $30° - 60° - 90°$ Triangles

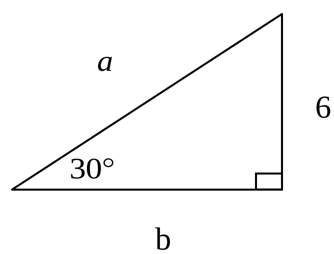
Given a $30° - 60° - 90°$ triangle with one side of length x, the relationship between the corresponding sides is:

$$1x : \sqrt{3}x : 2x$$

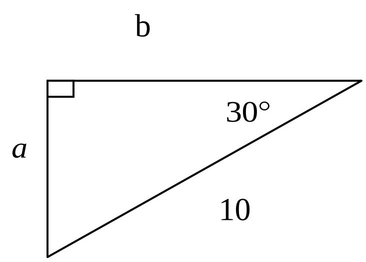


Example2. Find the missing site.

i)



ii)



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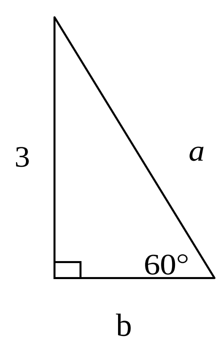
iii)



SECTION 4 SUPPLEMENTARY EXERCISES:

1. Evaluate.

$\sin 30° =$ $\qquad$ $\sin 60° =$

$\cos 30° =$ $\qquad$ $\cos 60° =$

$\tan 30° =$ $\qquad$ $\tan 60° =$

$\csc 30° =$ $\qquad$ $\csc 60° =$

$\sec 30° =$ $\qquad$ $\sec 60° =$

$\cot 30° =$ $\qquad$ $\cot 60° =$

$\sin 45° =$

$\cos 45° =$

$\tan 45° =$

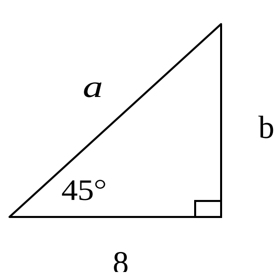
$\csc 45° =$

$\sec 45° =$

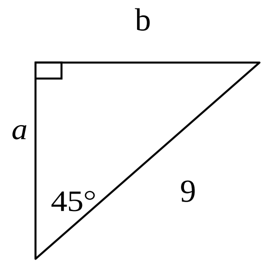
$\cot 45° =$

2. Find the missing sides.

i)



ii)



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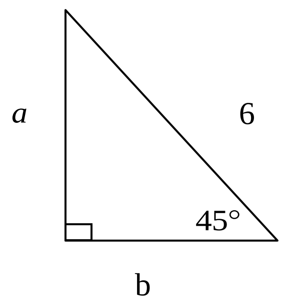
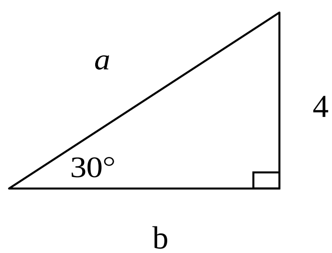
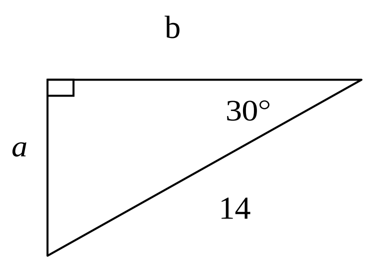
iii)



iv)



v)



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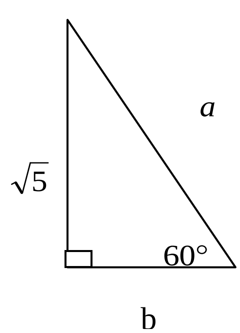
vi)



Section 5: Reference Angles

1. The **reference angle** for any angle θ in standard position is the positive acute angle between the terminal side of θ and the x-axis.

2. Quadrants

Quadrant I contains angles between ___________________

Quadrant II contains angles between ___________________

Quadrant III contains angles between ___________________

Quadrant IV contains angles between ___________________

Quadrant II

Positive functions:

Negative functions:

Quadrant I

Positive functions:

Negative functions:

Quadrant III

Positive functions:

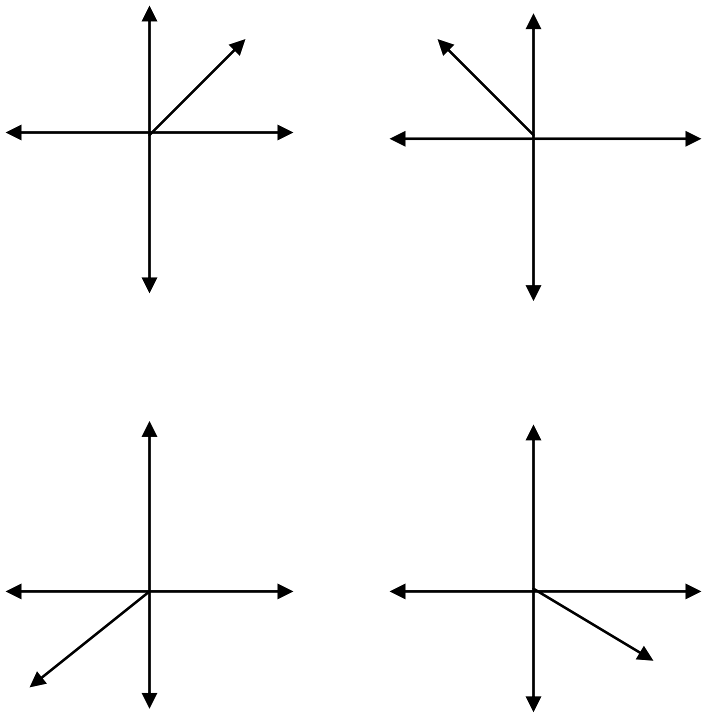
Negative functions:

Quadrant IV

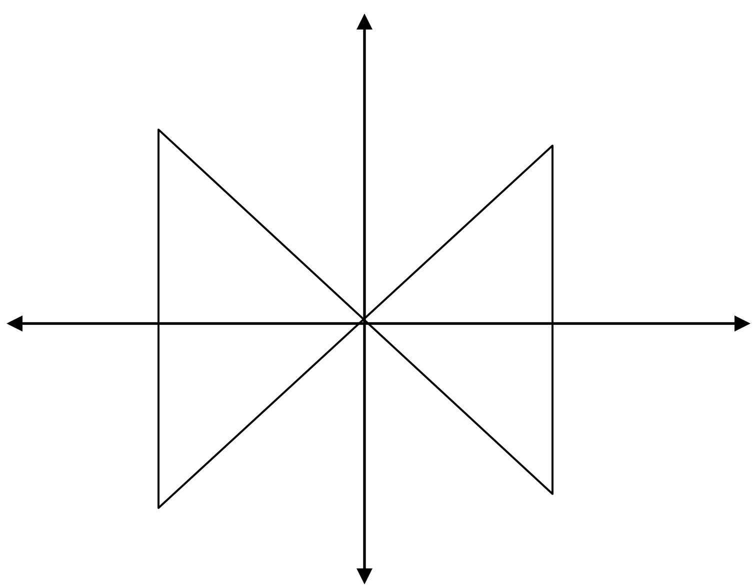
Positive functions:

Negative functions:

3. Draw where the reference angle is found:

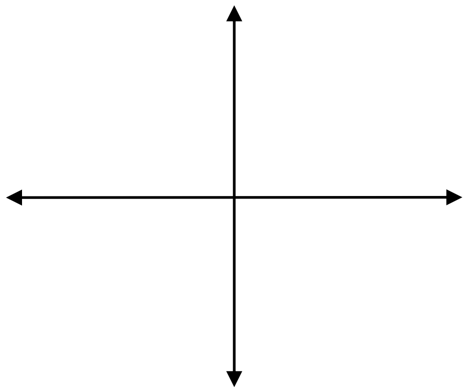


4. To find the reference angle of a triangle in the four quadrants draw a bow-tie.

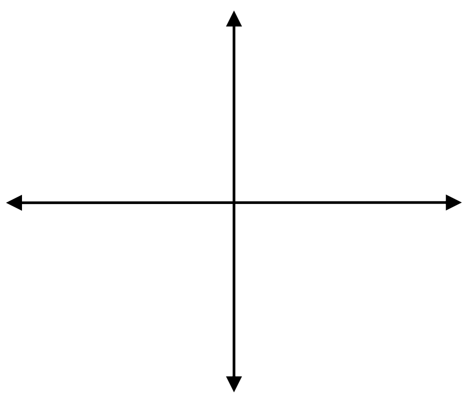


5. Example 1. Express the give trigonometric function in terms of the same function of a positive acute angle and find the value without using a calculator.

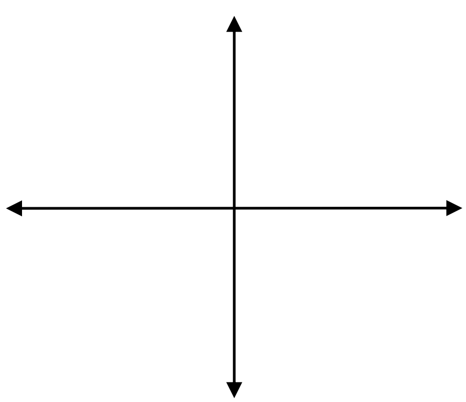
i) $\sin 150°$



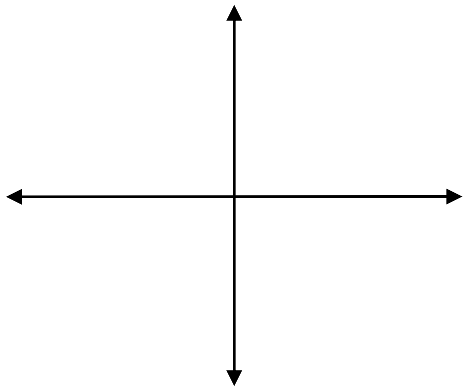
ii) $\tan 240°$



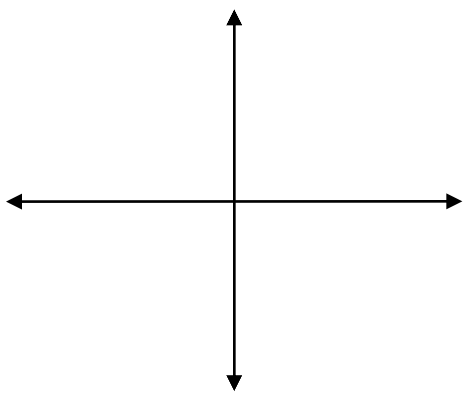
iii) $\cos 135°$



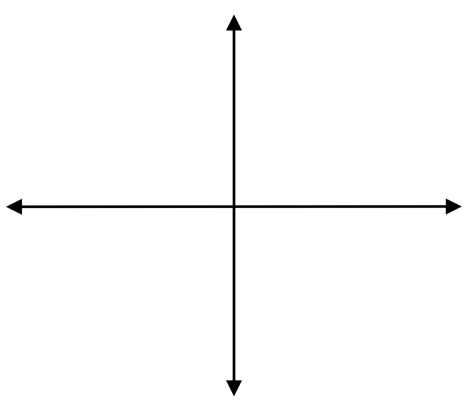
iv) $\tan 315°$



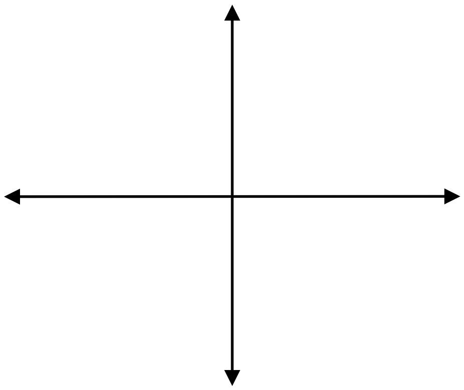
v) $\sin(-150°)$



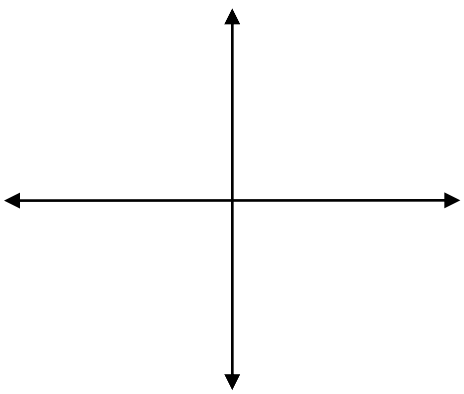
vi) $\cos(-210°)$



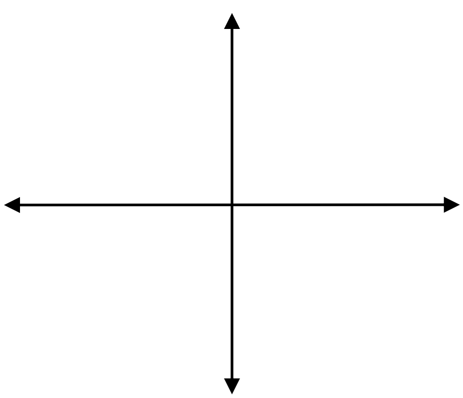
vii) $\tan(-135°)$



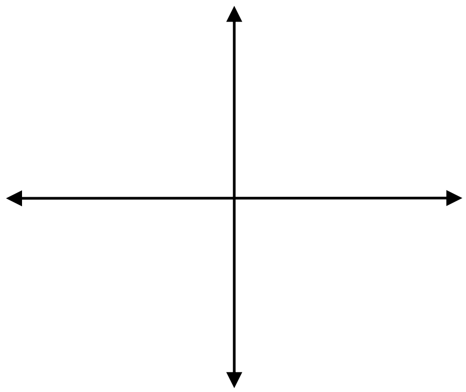
viii) $\sin 300°$



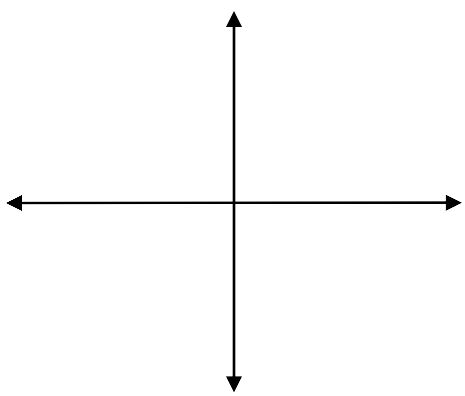
ix) $\sin(-45°)$



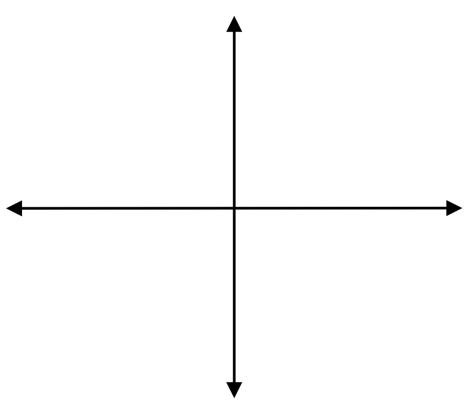
x) $\cos 495°$



xi) $\sin 390°$

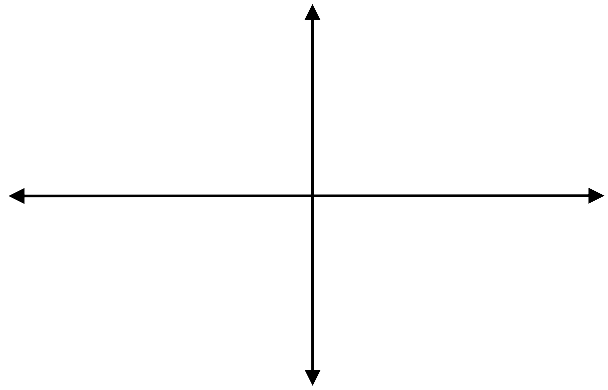


xii) $\tan 405°$



6. Example 2.

i) Find the values of the trigonometric functions of the angle θ with its terminal side passing through the point $(3,4)$:

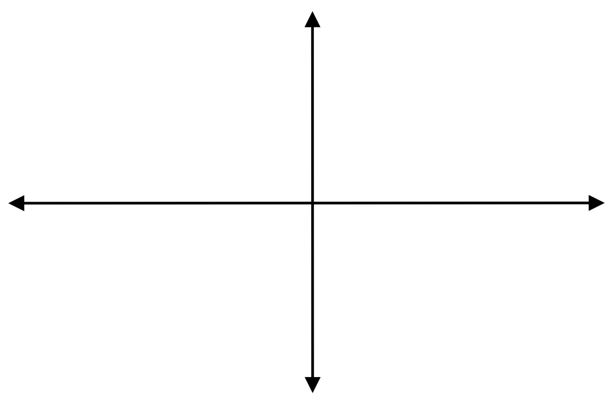


$\sin\theta =$ $\qquad\qquad$ $\csc\theta =$

$\cos\theta =$ $\qquad\qquad$ $\sec\theta =$

$\tan\theta =$ $\qquad\qquad$ $\cot\theta =$

ii) Find the values of the trigonometric functions of the angle θ with its terminal side passing through the point $(-5,12)$:

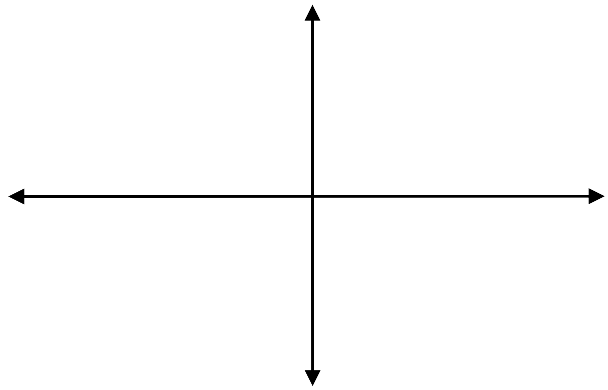


$\sin\theta =$ $\qquad\qquad$ $\csc\theta =$

$\cos\theta =$ $\qquad\qquad$ $\sec\theta =$

$\tan\theta =$ $\qquad\qquad$ $\cot\theta =$

iii) Find the values of the trigonometric functions of the angle θ with its terminal side passing through the point $(-7,-24)$:

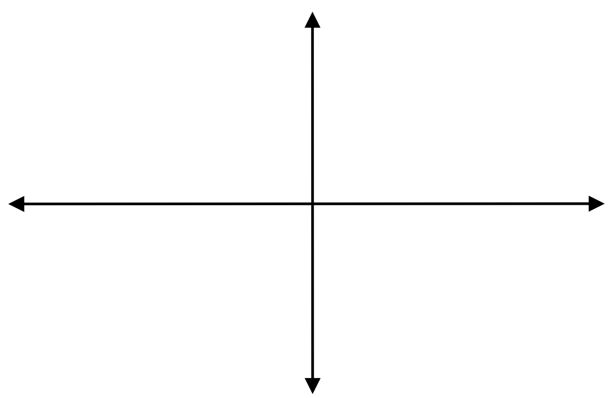


$\sin\theta =$ $\csc\theta =$

$\cos\theta =$ $\sec\theta =$

$\tan\theta =$ $\cot\theta =$

iv) Find the values of the trigonometric functions of the angle θ with its terminal side passing through the point $(6,-8)$:

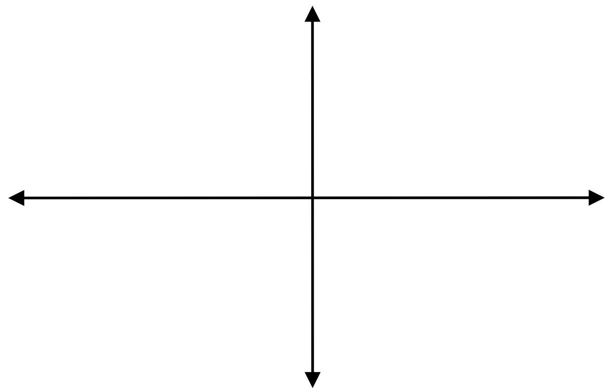


$\sin\theta =$ $\csc\theta =$

$\cos\theta =$ $\sec\theta =$

$\tan\theta =$ $\cot\theta =$

v) Find the values of the trigonometric functions of the angle θ with its terminal side passing through the point $(-3,3)$:

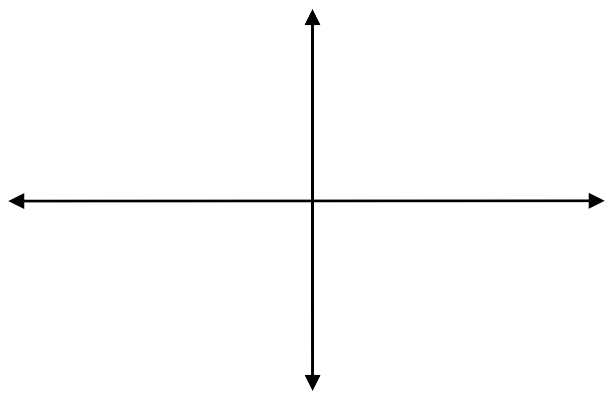


$\sin\theta =$ $\csc\theta =$

$\cos\theta =$ $\sec\theta =$

$\tan\theta =$ $\cot\theta =$

vi) Find the values of the trigonometric functions of the angle θ with its terminal side passing through the point $(-\sqrt{7},-\sqrt{3})$:

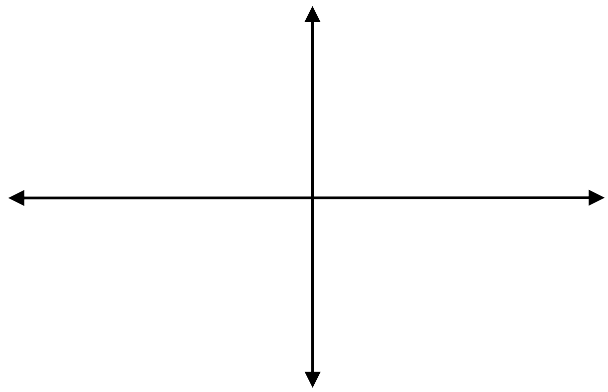


$\sin\theta =$ $\csc\theta =$

$\cos\theta =$ $\sec\theta =$

$\tan\theta =$ $\cot\theta =$

vii) Find the values of the trigonometric functions of the angle θ with its terminal side passing through the point $(-\sqrt{3},-5)$:



$\sin\theta =$ $\csc\theta =$

$\cos\theta =$ $\sec\theta =$

$\tan\theta =$ $\cot\theta =$

SECTION 5 SUPPLEMENTARY EXERCISES:

1. Fill in the blanks:
   a) $\sin\theta$ and $\csc\theta$ is positive in Quadrants _____ and _____
      $\sin\theta$ and $\csc\theta$ is negative in Quadrants _____ and _____

   b) $\cos\theta$ and $\sec\theta$ is positive in Quadrants _____ and _____
      $\cos\theta$ and $\sec\theta$ is negative in Quadrants _____ and _____

   c) $\tan\theta$ and $\cot\theta$ is positive in Quadrants _____ and _____
      $\tan\theta$ and $\cot\theta$ is negative in Quadrants _____ and _____

2. Express the give trigonometric function in terms of the same function of a positive acute angle and find the value without using a calculator.

   a) $\cos 150°$
   b) $\sin 240°$
   c) $\tan 225°$
   d) $\cos 300°$
   e) $\tan 120°$
   f) $\sin 135°$
   g) $\cos(-240°)$
   h) $\tan(-150°)$
   i) $\cot(-135^\circ)$
   j) $\sin(480^\circ)$
   k) $\cos(210^\circ)$
   l) $\cos(-225^\circ)$

3. Find the values of the trigonometric functions of the angle θ with its terminal side passing through the following points:
   a) $(1,1)$
   b) $(-4,3)$
   c) $(\sqrt{5},-2)$
   d) $(6,-\sqrt{10})$
   e) $(-\sqrt{6},-\sqrt{7})$
   f) $(\sqrt{3},\sqrt{6})$
   g) $(-5,-12)$
   h) $(\sqrt{7},-12)$
   i) $(\frac{5}{2},7)$
   j) $(-3\sqrt{3},\sqrt{5})$
   k) $(-3,-11)$

Section 6: Solving Trigonometric Equations

1. In algebra, linear equations are solved by isolating the variable and quadratic equations by factoring.

Example 1. Solve for x.

i) $2x - 1 = 0$

ii) $\sqrt{3}\,x - 1 = 0$

iii) $4x^2 - 1 = 0$

2. The process of solving trigonometric equations is very similar to the process of solving algebraic equations. With trigonometric equations, we look for values of an angle by solving for a specific trigonometric function of that angle.

Example: Find all solutions of the following equations in degrees in the interval $[0°, 360°)$ and in radians in the interval $[0, 2\pi)$

i) Solve: $2\sin x - 1 = 0$

ii) Solve: $2\cos x + 1 = 0$

iii) Solve: $5\tan x + 5 = 0$

iv) Solve: $\sqrt{3}\cos x - 1 = 0$

v) Solve: $5\sin x - \sqrt{3} = 3\sin x$

vi) Solve: $2\cos x = 6\cos x - \sqrt{12}$

vii) Solve: $4\sin^2 x - 1 = 0$

viii) Solve: $6\tan^2 x - 6 = 0$

ix) Solve: $(2\cos x - \sqrt{3})(2\cos x - 1) = 0$

SECTION 6 SUPPLEMENTARY EXERCISES:

1. Find all solutions of the following equations in degrees in the interval $[0°, 360°)$ and in radians in the interval $[0, 2\pi)$

   a) Solve: $2\cos x - 1 = 0$

   b) Solve: $2\sin x + 1 = 0$

   c) Solve: $7\tan x - 7 = 0$

   d) Solve: $\sqrt{3}\tan x - 1 = 0$

   e) Solve: $5\cos x - \sqrt{3} = 3\cos x$

   f) Solve: $2\sin x = 6\sin x - \sqrt{12}$

   g) Solve: $4\cos^2 x - 1 = 0$

   h) Solve: $5\tan^2 x - 5 = 0$

   i) Solve: $(2\sin x - \sqrt{3})(2\sin x - 1) = 0$

   j) Solve: $3(\sin x + 2) = 3 - \sin x$

   k) Solve: $(3\tan x + 1)(\tan x - 2) = 0$

   l) Solve: $4(\cot x + 1) = 2(\cot x + 2)$

   m) Solve: $3\cos^2 x - 4\cos x + 1 = 0$

   n) Solve: $3\sin^2 x + 7\sin x + 2 = 0$

   o) Solve: $2\cot^2 x - 13\cot x + 6 = 0$

Section 7: Trigonometric Identities

1. Reciprocal Identities

$$\sin\theta = \frac{1}{\csc\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

$$\cos\theta = \frac{1}{\sec\theta} \qquad \sec\theta = \frac{1}{\cos\theta}$$

$$\tan\theta = \frac{1}{\cot\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

2. Ratio Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

3. Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1$$

$\sin^2\theta =$ ______________________________ = (   )(   )

$\cos^2\theta =$ ______________________________ = (   )(   )

$$1 + \cot^2\theta = \csc^2\theta$$

$\cot^2\theta =$ ______________________________ = (   )(   )

$1 =$ ___________________________________ = (   )(   )

$$\tan^2\theta + 1 = \sec^2\theta$$

$\tan^2\theta =$ ______________________________ = (   )(   )

$1 =$ ___________________________________ = (   )(   )

4. i) Derivation of the Pythagorean Identity $1+\cot^2\theta=\csc^2\theta$

Begin with $\sin^2\theta+\cos^2\theta=1$

Divide the equation by $\sin^2\theta$.

ii) Derivation of the Pythagorean Identity $\tan^2\theta+1=\sec^2\theta$

Begin with $\sin^2\theta+\cos^2\theta=1$

Divide the equation by $\cos^2\theta$.

5. **To prove or verify a trigonometric identity**, we use trigonometric substitution and algebraic manipulations to either
   a) transform the right side of the identity into the left side, or
   b) transform the left side of the identity into the right side.

Example1.

i) Prove: $\cos\theta\tan\theta = \sin\theta$

ii) Prove: $\cot\alpha\sec\alpha\sin\alpha = 1$

iii) Prove: $\sec x \csc x = \cot x + \tan x$

iv) Prove: $\dfrac{\cos y}{1-\sin y} = \dfrac{1+\sin y}{\cos y}$

v) Prove: $1+\cos A=\dfrac{\sin^2 A}{1-\cos A}$

vi) Prove: $\tan B+\cot B=\sec B\csc B$

vii) Prove: $\sin^2 a = \dfrac{1-\cos^4 a}{1+\cos^2 a}$

SECTION 7 SUPPLEMENTARY EXERCISES:

1. Prove the following identities:

a) $\dfrac{\tan A}{\sin A} = \sec A$

b) $\dfrac{\cot \theta}{\csc \theta} = \cos \theta$

c) $\sin B \cot^2 B + \sin B = \csc B$

d) $\dfrac{1+\cos y}{\sin y} = \dfrac{\sin y}{1-\cos y}$

e) $\dfrac{1+\tan x}{1+\cot x} = \tan x$

f) $\dfrac{1}{\cos B} - \cos B = \sin B \tan B$

g) $2\csc^2 t = \dfrac{1}{1-\cos t} + \dfrac{1}{1+\cos t}$

h) $\sin^4 x - \cos^4 x = 2\sin^2 x - 1$

i) $\sin x \cos x \tan x = 1 - \cos^2 x$

j) $$\frac{\tan x}{\sec x}+\frac{\cot x}{\csc x}=\sin x+\cos x$$

k) $$\frac{\cos x+1}{\sin^3 x}=\frac{\csc x}{1-\cos x}$$

l) $$\cos x+\tan x=\frac{\cos^2 x+\sin x}{\cos x}$$

m) $$\cot x+\sin x=\frac{\cos x+\sin^2 x}{\sin x}$$

n) $$\frac{\tan x}{1+\tan x}=\frac{\sin x}{\sin x+\cos x}$$

p) $$\frac{1+\cos x}{\tan x+\sin x}=\cot x$$

q) $$\frac{\tan x-\sin x}{\tan x\sin x}=\frac{1-\cos x}{\sin x}$$