Helpful Radical Multiplication Rule:

Square of a Radical: $(\sqrt{a})^2 = (\sqrt{a})(\sqrt{a}) = a$

Product of radical conjugates: $(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b}) = (\sqrt{a})^2 - (\sqrt{b})^2 = a - b$

Or $(\sqrt{a}+b)(\sqrt{a}-b) = (\sqrt{a})^2 - (b)^2 = a - b^2$

Dividing Radicals by Rationalizing the Denominators

Strategy for rationalizing denominators with radicals:

If the denominator has one term, multiply the numerator and the denominator by the radical:

Examples a) $\frac{2}{\sqrt{a}} = \frac{2}{\sqrt{a}} \cdot \frac{\sqrt{a}}{\sqrt{a}} = \frac{2\sqrt{a}}{a}$

$\frac{y}{\sqrt{x}} = \frac{y\sqrt{x}}{x}$

b) $\frac{4}{3\sqrt{a}} = \frac{4}{3\sqrt{a}} \cdot \frac{\sqrt{a}}{\sqrt{a}} = \frac{4\sqrt{a}}{3a}$

If the denominator has two terms, multiply the numerator and the denominator by the conjugate of the denominator:

Examples a) $\frac{3}{\sqrt{a}+\sqrt{b}} = \frac{3}{\sqrt{a}+\sqrt{b}} \cdot \frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}-\sqrt{b}} = \frac{3(\sqrt{a}-\sqrt{b})}{a-b}$

$x+y$ and $x-y$ are conjugates

$(x+y)(x-y) = x^2 - y^2$

b) $\frac{5}{\sqrt{a}-b} = \frac{5}{\sqrt{a}-b} \cdot \frac{\sqrt{a}+b}{\sqrt{a}+b} = \frac{5(\sqrt{a}+b)}{a-b^2}$

Exercise 8: Rationalize denominators. Simplify the answer.

a) $\frac{9}{\sqrt{7}} \cdot \frac{\sqrt{7}}{\sqrt{7}} = \frac{9\sqrt{7}}{\sqrt{49}} = \frac{9\sqrt{7}}{7}$

b) $\frac{3}{2\sqrt{x}} \cdot \frac{\sqrt{x}}{\sqrt{x}} = \frac{3\sqrt{x}}{2\sqrt{x^2}} = \boxed{\frac{3\sqrt{x}}{2x}}$

c) $\frac{-12}{\sqrt{10}-\sqrt{6}} \cdot \frac{\sqrt{10}+\sqrt{6}}{\sqrt{10}+\sqrt{6}}$

$\frac{-12(\sqrt{10}+\sqrt{6})}{(\sqrt{10}-\sqrt{6})(\sqrt{10}+\sqrt{6})}$

$\frac{-12(\sqrt{10}+\sqrt{6})}{(\sqrt{10})^2-(\sqrt{6})^2}$

$\frac{-12(\sqrt{10}+\sqrt{6})}{10-6}$

$= \frac{-12(\sqrt{10}+\sqrt{6})}{4}$

$= \boxed{-3(\sqrt{10}+\sqrt{6})}$

d) $\left(\frac{\sqrt{6}}{5-\sqrt{2}}\right) \cdot \left(\frac{5+\sqrt{2}}{5+\sqrt{2}}\right)$

$\frac{\sqrt{6}(5+\sqrt{2})}{(5-\sqrt{2})(5+\sqrt{2})}$

$(a-b)(a+b) = a^2 - b^2$

$\frac{5\sqrt{6}+\sqrt{12}}{25-(\sqrt{2})^2}$

$\frac{5\sqrt{6}+\sqrt{4}\sqrt{3}}{25-2}$

$\boxed{\frac{5\sqrt{6}+2\sqrt{3}}{23}}$

SECTION 2.1 SUPPLEMENTARY EXERCISES

1. Simplify the expressions.

   a) $\sqrt{48}$

   b) $\sqrt{20} \cdot \sqrt{5}$

   c) $\sqrt{\frac{64}{25}}$

   d) $\sqrt{\frac{13}{49}}$

2. Rationalizing the denominator.

   a) $\sqrt{\frac{5}{6}}$

   b) $\frac{9}{\sqrt{2}}$

   c) $\sqrt{\frac{36}{7}}$

   d) $\frac{5}{\sqrt{2}-\sqrt{3}}$

   d) $\frac{\sqrt{7}}{\sqrt{7}+4}$

   d) $\frac{\sqrt{x}+\sqrt{3}}{\sqrt{x}-\sqrt{3}}$

Section 2.2 Solving Quadratic Equations by the Square Root Property

The Square Root Property

If $x^2 = a$ and $a \geq 0$, then $x = \sqrt{a}$ or $x = -\sqrt{a}$.

Sometimes it is indicated as $x = \pm\sqrt{a}$

Solving a Quadratic Equation by the Square Root Property

Isolate the square and then apply the Square Root Property

Exercise 1: Solve using the square root Property. Simplify the radical. Write out both solutions.

a) $x^2 = 9$

$\sqrt{x^2} = \pm\sqrt{9}$

$x = \pm 3$

b) $x^2 - 54 = 0$

$+54 \quad +54$

$x^2 = 54$

$\sqrt{x^2} = \pm\sqrt{54}$

$x = \pm\sqrt{9}\sqrt{6}$

$x = \pm 3\sqrt{6}$

Exercise 2: Solve using the square root Property. Simplify the radical. Write out both solutions.

a) $(a-5)^2 = 16$

$\sqrt{(a-5)^2} = \pm\sqrt{16}$

$a - 5 = \pm 4$

$a-5=-4$ | $a-5=4$

$+5 \quad +5$ | $+5 \quad +5$

$a = 1$ | $a = 9$

b) $(4t-3)^2 - 18 = 0$

$+18 \quad +18$

$(4t-3)^2 = 18$

$\sqrt{(4t-3)^2} = \pm\sqrt{18}$

$4t - 3 = \pm 3\sqrt{2}$

$+3 \quad +3$

$\frac{4t}{4} = \frac{3 \pm 3\sqrt{2}}{4}$

$\boxed{t = \frac{3 \pm 3\sqrt{2}}{4}}$

Section 2.3 Solving Quadratic Equations by Completing the Square

To solve a quadratic equation of the form $ax^2 + bx + c = 0$ by completing the square, we first divide the equation by the leading coefficient a. The procedure below will assume $a = 1$ and the quadratic equation is of the form $x^2 + bx + c = 0$

Solving a Quadratic Equation of the Form $x^2 + bx + c = 0$ by Completing the Square:

Steps
1. Write the left side in the form $x^2 + bx$ and move the constant term to the right side.
2. Add a constant, $\left(\frac{b}{2}\right)^2$, that will complete the square. The constant is added to both sides of the equation.
3. Write the left side as a binomial squared.
4. Apply the Square Root Property.
5. Solve for x and simplify.

Example: Solve by completing the square: $x^2 + 8x + 4 = 0$

Step 1: Write the left side in the form $x^2 + bx$ and move the constant term to the right side.	$x^2 + 8x = -4$
Step 2: Add a constant $\left(\frac{8}{2}\right)^2 = \mathbf{16}$ to both sides of the equation.	$x^2 + 8x + \mathbf{16} = -4 + \mathbf{16}$ $x^2 + bx + \left(\frac{b}{2}\right)^2$
Step 3: Write the left side as a binomial squared.	$(x + 4)^2 = 12$ $\left(x + \frac{b}{2}\right)^2$
Step 4: Apply the Square Root Property.	$\sqrt{(x+4)^2} = \pm\sqrt{12}$ $x + 4 = \pm\sqrt{12}$
Step 5: Solve for x and simplify.	$x = -4 \pm \sqrt{12}$ $x = -4 \pm 2\sqrt{3}$

Exercise 1: Solve by completing the square. $y^2 - 6y - 8 = 0$

$\boxed{y = 3 \pm \sqrt{17}}$

Complete the square on both sides of the equation

$b = -6$ $\left(\frac{b}{2}\right)^2 = \left(\frac{-6}{2}\right)^2 = (-3)^2 = 9$

$y^2 - 6y = 8$
$y^2 - 6y + 9 = 8 + 9$
$(y-3)^2 = 17$
$y - 3 = \pm\sqrt{17}$

$y^2 - 6x + 9$ $*a = 1$
$(y-3)(y-3)$ $b = -6 = -3 + -3$
$(y-3)^2$ $c = 9 = -3 \cdot -3$

Exercise 2: Solve by completing the square. $2w^2 + 12w - 14 = 0$

$2(w^2 + 6w - 7) = 0$

$b = 6$
$\frac{b}{2} = +3$
$\left(\frac{b}{2}\right)^2 = 9$

$2 \neq 0$

$w^2 + 6w - 7 = 0$
$w^2 + 6w + 9 = 7 + 9$
$(w+3)^2 = 16$

$\sqrt{(w+3)^2} = \pm\sqrt{16}$
$w + 3 = \pm 4$
$-3 \quad -3$
$w = -3 \pm 4$
$w = -3 - 4$, $w = -3 + 4$
$\boxed{w = -7} \; \boxed{w = 1}$

$\frac{2w^2 + 12w}{2} = \frac{14}{2}$
$w^2 + 6w = 7$
$\vdots$

SECTION 2.2 SUPPLEMENTARY EXERCISES

1. Solve the following equations by using the square root Property:

a. $x^2 = 100$

b. $s^2 - 12 = 0$

c. $y^2 + 7 = 88$

d. $(y-3)^2 - 16 = 0$

e. $(y+7)^2 - 25 = 0$

f. $(y-1)^2 - 12 = 0$

g. $(2k-1)^2 - 9 = 0$

f. $9(2m-3)^2 + 8 = 449$

SECTION 2.3 SUPPLEMENTARY EXERCISES

1. Solve the following equations by completing the square:

a. $p^2 - p = 6$

b. $x^2 + 10x - 7 = 0$

c. $y^2 - 3y - 6 = 0$

d. $6k^2 + 17k + 5 = 0$

e. $5m^2 - 10m + 3 = 0$

f. $3x^2 + 4 = 8x$

g. $6w^2 + 12x = 48$

h. $x^2 - 10x + 26 = 8$

Module III

Quadratic Equations and Graphs

The ancient Egyptians were interested in the solutions of a quadratic equation in order to determine the dimension of a floor plan for a desired area and perimeter of a given shape. If a rectangular room, for example, they would be interested in the solutions to

$$L \cdot W = A$$
$$2L + 2W = P$$

which we can now solve algebraically with a quadratic equation. Without the sophisticated number system nor the method of quadratic equations, the ancient Egyptians developed a lookup table of standard dimensions and sizes for different shapes. Engineers would find the most fitting design based on the table developed. Unfortunately, there were limitations using the table and errors from incoherent and erroneous reproduction of the tables, which led to flawed engineering.

Hell, D. (2004). History behind Quadratic Equations. Retrieved on June 22,2011.

Section 3.1: Solving Quadratic Equations by the Quadratic Formula

The quadratic $ax^2 + bx + c = 0$ can be solved in many ways:

I. Factor the quadratic and then use the Zero Factor Theorem (Section 1.2)
2. The Square root property (Section 2.2)
3. Completing the square (Section 2.3)
4. The Quadratic Formula (Section 3.1)

The Quadratic Formula

A quadratic equation of the form $ax^2 + bx + c = 0$, where $a \neq 0$, has two solutions:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Exercise 1: Solve using the quadratic formula

a) $x^2 - 5x - 8 = 0$

$a=1$

$b=-5$

$c=-8$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(-8)}}{2(1)}$$

$$x = \frac{5 \pm \sqrt{25+32}}{2}$$

$$x = \frac{5 \pm \sqrt{57}}{2}$$

b) $x^2 + 6x + 2 = 0$

$$x = \frac{-(6) \pm \sqrt{(6)^2 - 4(1)(2)}}{2(1)}$$

$$= \frac{-6 \pm \sqrt{36-8}}{2}$$

$$= \frac{-6 \pm \sqrt{28}}{2}$$

$$= \frac{-6 \pm \sqrt{4}\sqrt{7}}{2}$$

$$= \frac{-6 \pm 2\sqrt{7}}{2}$$

$$= \boxed{-3 \pm \sqrt{7}}$$

c) $2x^2 - 4x = 3$

$2x^2 - 4x - 3 = 0$

need $ax^2+bx+c=0$ to use q.f.

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(2)(-3)}}{2(2)}$$

$$= \frac{4 \pm \sqrt{16+24}}{4}$$

$$= \frac{4 \pm \sqrt{40}}{4}$$

$$= \frac{4 \pm \sqrt{4}\sqrt{10}}{4}$$

$$= \frac{4 \pm 2\sqrt{10}}{4} = \frac{\cancel{2}(2+\sqrt{10})}{\cancel{2}(2)}$$

$$= \boxed{1 \pm \frac{\sqrt{10}}{2}} \qquad = \boxed{\frac{2+\sqrt{10}}{2}}$$

<u>**How many errors can you find?**</u>

Find where the errors occur and provide the correct solution

Solve using the quadratic formula $x^2 - 4x = 16$

Find the errors (at least six)

$a = 1,\ b = -4,\ c = 16$

$$x = \frac{-4 \pm \sqrt{4^2 - 4(1)(16)}}{2(1)}$$

$$= \frac{-4 \pm \sqrt{16 - 64}}{2}$$

$$= \frac{-4 \pm \sqrt{-48}}{2}$$

$$= -4 \pm \sqrt{-24}$$

$$= -4 \pm 2\sqrt{6}$$

The two solutions are:

$x = -2\sqrt{6}$ or $x = -6\sqrt{6}$

Provide the correct solution here

Exercise 2: Which is the best method for solving $4x^2 - 4x = 0$?
(a) by factoring; (b) the square root property; (c) completing the square; (d) quadratic formula.
Show solution:

$$4x(x-1)=0$$
$$4x=0 \quad | \quad x-1=0$$
$$\boxed{x=0 \quad | \quad x=1}$$

$$x=\frac{-(-4)\pm\sqrt{(-4)^2-4(4)(0)}}{2(4)}$$
$$x=\frac{4\pm\sqrt{16}}{8}$$
$$x=\frac{4\pm 4}{8}$$
$$x=\frac{4+4}{8}=1$$
$$x=\frac{4-4}{8}=0$$

Exercise 3: Which is the best method for solving $x^2 - 63 = 0$?
(a) by factoring; (b) the square root property; (c) completing the square; (d) quadratic formula.
Show solution:

$$x^2=63$$
$$x=\pm\sqrt{63}$$
$$x=\pm\sqrt{9}\sqrt{7}$$
$$x=\pm 3\sqrt{7}$$

Exercise 4: Which is the best method for solving $x^2 - 6x - 9 = 0$?
(a) by factoring; (b) the square root property; (c) completing the square; (d) quadratic formula.
Show solution:

$b=-6$

$\frac{b}{2}=-3$

$\left(\frac{b}{2}\right)^2=9$

$$x^2-6x-9=0$$
$$x^2-6x \quad =9$$
$$x^2-6x+9=9+9$$
$$(x-3)^2=18$$
$$x-3=\pm 3\sqrt{2}$$ ← square root both sides

$$\boxed{x=3\pm 3\sqrt{2}}$$

Exercise 5: Which is the best method for solving $2x^2 - 6x - 9 = 0$?
(a) by factoring; (b) the square root property; (c) completing the square; (d) quadratic formula.
Show solution:

$$2x^2-6x \quad =9$$
$$\frac{2x^2-6x}{2}=\frac{9}{2}$$
$$x^2-3x=\frac{9}{2}$$
$$x^2-3x+\frac{9}{4}=\frac{\cancel{9}^{18}}{\cancel{2}_4}+\frac{9}{4}$$
$$\left(x-\frac{3}{2}\right)^2=\frac{27}{4}$$

$b=-3$

$\frac{b}{2}=-\frac{3}{2}$

$\left(\frac{b}{2}\right)^2=\frac{9}{4}$

$$x-\frac{3}{2}=\pm\sqrt{\frac{27}{4}}$$
$$x-\frac{3}{2}=\pm\frac{3\sqrt{3}}{2}$$
$$\boxed{x=\frac{3}{2}\pm\frac{3\sqrt{3}}{2}}$$
$$\boxed{x=\frac{3\pm 3\sqrt{3}}{2}}$$

SECTION 3.1 SUPPLEMENTARY EXERCISES

Solve the following equations by using the quadratic formula:

1. $y^2 + 10y + 9 = 0$
2. $s^2 - s = 30$
3. $m^2 - 81 = 0$
4. $6x^2 + 2x + 3 = 0$
5. $10x^2 - 13x - 3 = 0$
6. $4x^2 + 6x + 1 = 0$
7. $m^2 + 5m = 2$
8. $2k^2 + 9k = -7$
9. $(z-2)(z+4) + 6 = 0$
10. $(3x-7)(x+5) = -31$
11. $11n^2 - 4n(n-2) = 6(n+3)$
12. $x(x-3) = -10x - 7$

Section 3.2 Graph of Quadratic Equations - Parabolas

The graph of a **quadratic equation of** the form $y = ax^2 + bx + c$ where $a \neq 0$ is a parabola (U-shaped). The parabola has either a maximum or a minimum point, called the **vertex**. Often times, it is more convenient to express the quadratic equation in the vertex form.

Method 1: The Vertex Form

The Vertex Form of a Quadratic equation is $y = a(x-h)^2 + k$, where (h, k) is the vertex.

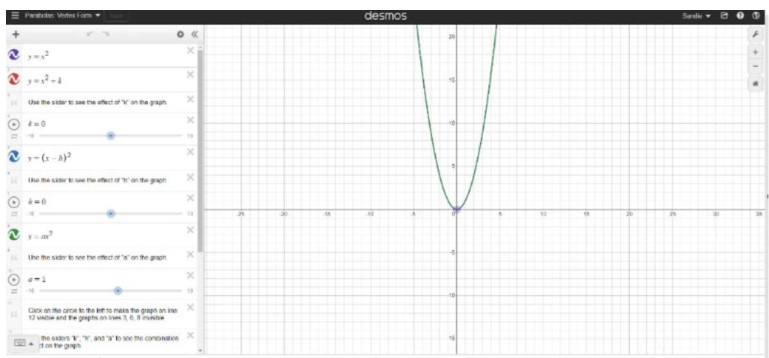
Method 2: The Vertex Formula

The Vertex Formula of a quadratic equation of the form $y = ax^2 + bx + c$:

x-coordinate of the vertex is $-\frac{b}{2a}$

Substitute x-coordinate to find the y-coordinate of the vertex

Use the Desmos link below to see the effect of changing h, k and a on the parabola.
https://www.desmos.com/calculator/0txid19ts5



How does the graph change when you change h? When h is positive? When h is negative?

if h is positive the graph moves to the right
" negative " " left.

How does the graph change when you change k? When k is positive? When k is negative?

if k is positive graph moves ↑
negative ↓

How does the graph change when you change a? When a is positive? When a is negative? When $|a| > 1$? When $|a| < 1$?

a is positive ⇒ "normal" a is negative ⇒ "inverted
if $|a| > 1$ sharper graph
if $0 < |a| < 1$ wider graph.

$0^2 \quad 1^2 \quad 2^2 \quad 3^2 \quad 4^2$

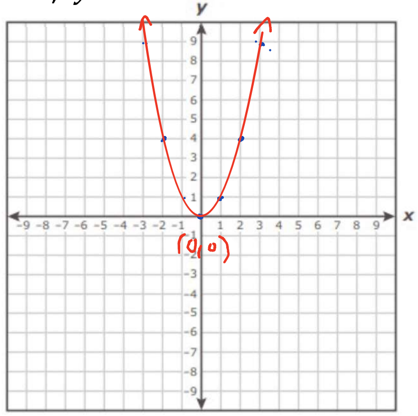
$0 \quad 1 \quad 4 \quad 9 \quad 16$

$1 \quad 3 \quad 5 \quad 7$

Exercise 1: For each function: (i) Identify the vertex; (ii) sketch the graph.

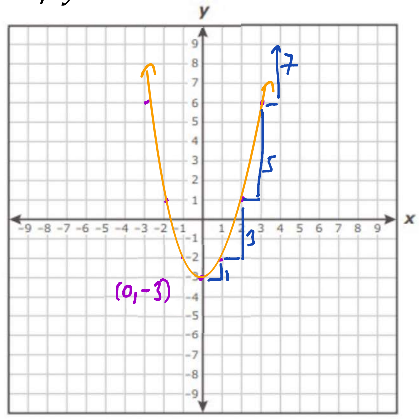
$y=a(x-h)^2+k$, (h,k) is vertex

a) $y = x^2$



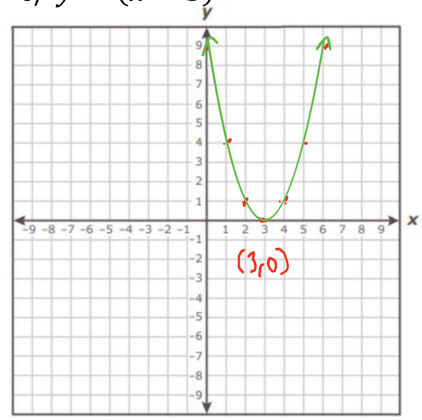
The vertex is $(0,0)$

b) $y = x^2 - 3$



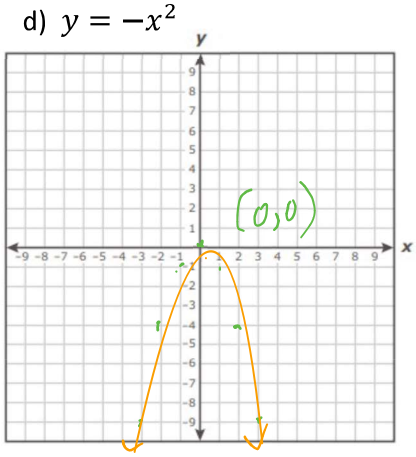
The vertex is $(0,-3)$

c) $y = (x-3)^2$



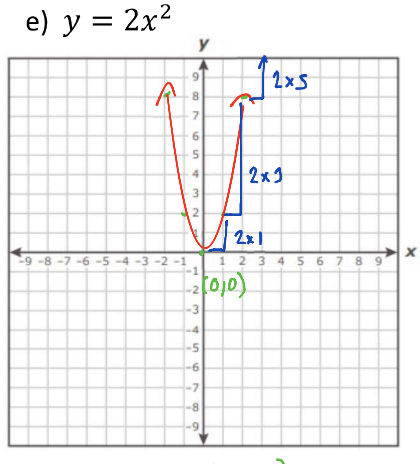
The vertex is $(3,0)$

d) $y = -x^2$



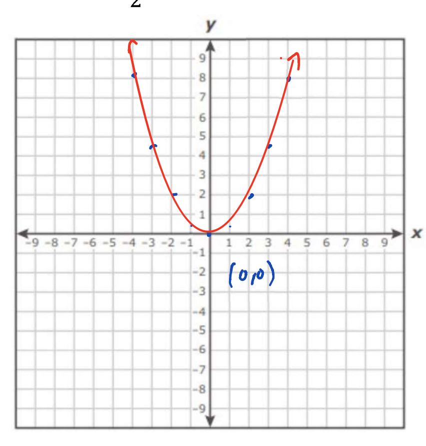
The vertex is $(0,0)$

e) $y = 2x^2$



The vertex is $(0,0)$

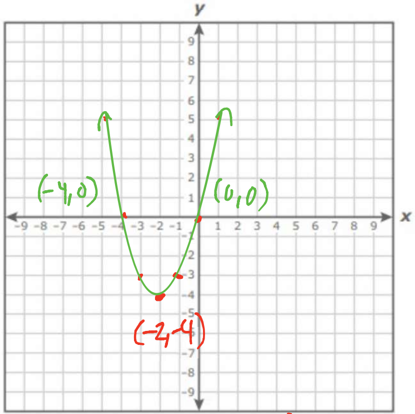
f) $y = \frac{1}{2}x^2$



The vertex is $(0,0)$

Exercise 2: Use the **Vertex Form** to identify the vertex. Sketch the graph. Find the roots (solutions) of the equation by setting $y = 0$ and solve.

a) $y = (x+2)^2 - 4 = (x-(-2))^2 - 4$ $\quad a=1$



roots

$(x+2)^2 - 4 = 0$

$(x+2)^2 = 4$

$x+2 = \pm 2$

$x = -2 \pm 2$

$x = 0$ or $x = -4$

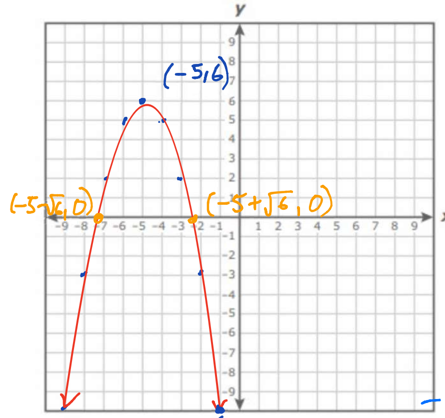
The vertex is $(-2,-4)$

The roots are $x=0$, $x=-4$

b) $y = -(x+5)^2 + 6 \Rightarrow y = -(x-(-5))^2 + 6$

$a = -1$

$\rightarrow$ reflected



roots

$0 = -(x+5)^2 + 6$

$-6 = -(x+5)^2$

$6 = (x+5)^2$

$\pm\sqrt{6} = x+5$

$-5 \pm \sqrt{6} = x$

The vertex is $(-5,6)$

The roots are $x = -5-\sqrt{6}$, $x = -5+\sqrt{6}$

Exercise 3: Use the **Vertex Formula** to find the vertex. Sketch the graph. Find the roots (solutions) of the equation by setting $y = 0$ and solve.

vertex $h = -\frac{b}{2a}$ $\quad k = ah^2 + bh + c$

a) $y = x^2 + 4x - 5$ $\quad a=1,\ b=4\ \ c=-5$

standard form

* if $y = ax^2+bx+c$ is in standard form

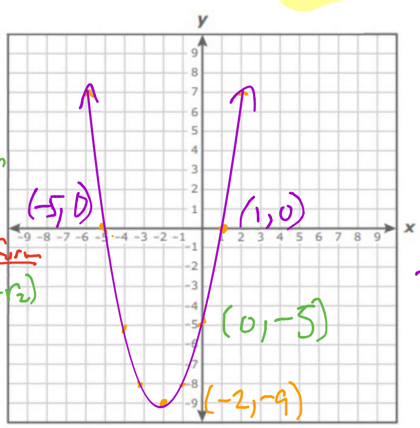
y-int $(0,c)$

in factored form

* if $y = a(x-r_1)(x-r_2)$

x-int

$(r_1,0), (r_2,0)$



vertex

$h = \frac{-b}{2a} = -\frac{4}{2(1)} = -2$

$k = (-2)^2 + 4(-2) - 5$

$= 4 - 8 - 5 = -9$

$(h,k) = (-2,-9)$

roots

$x^2 + 4x - 5 = 0$

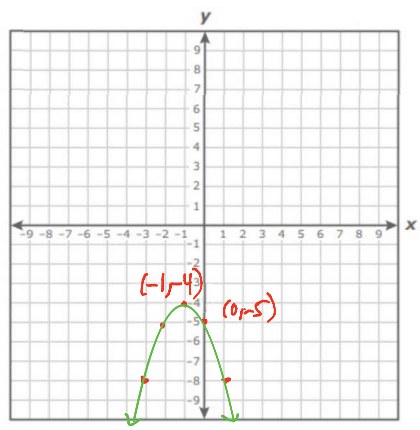
$(x+5)(x-1) = 0$

$x = -5 \quad x = 1$

The vertex is $(-2,-9)$

The roots are -5 , 1

b) $y = -x^2 - 2x - 5$



vertex

$h = -\frac{b}{2a} = -\frac{(-2)}{2(-1)} = -1$

$k = -(-1)^2 - 2(-1) - 5$

$= -1 + 2 - 5$

$= -4$

roots

$-x^2 - 2x - 5 = 0$

$x^2 + 2x + 5 = 0$

$x = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(5)}}{2(1)}$

$x = \frac{-2 \pm \sqrt{-16}}{2}$

$x = \frac{-2 \pm 4i}{2}$

$x = -1 + 2i$

The vertex is $(-1,-4)$

The roots are imaginary