1 Rational exponents and roots

1.1 Prelude

Recall that $2^4 = 2 \cdot 2 \cdot 2 \cdot 2$, that is, the exponent counts the number of times the base (in this case 2) gets **multiplied**. This is similar to the expression $4 \cdot 2$ where the 4 counts the number of 2's that are **added**.
It follows that for example:

$$3^2 \cdot 3^4 = \qquad\qquad\qquad \text{(write it out)} = 3^?.$$

$$(4^3)^2 = \qquad\qquad\qquad \text{(write it out)} = 4^?.$$

What are the general rules?

$$a^b a^c = ?$$

$$(a^b)^c = ?$$

Write out

$$(5 \cdot 2)^3 = 5^? \cdot 2^?$$

Can you use this to find the general rule

$$(ab)^c =$$

Write out

$$\left(\frac{5}{2}\right)^3 = \frac{5^?}{2^?}$$

Can you use this to find the general rule

$$\left(\frac{a}{b}\right)^c =$$

1.2 Non-natural number exponents

Consider the expression a^0. None of the rules we have discussed so far make sense here.

We would like to give it a meaning so that the rules of exponents continue to work!

Consider $a^1 \cdot a^0$ assuming the rules above are valid. Then

$$a^1 \cdot a^0 = a^{1+0} = a^1 \text{using the appropriate rule}$$

So $a \cdot a^0 = a$. Dividing both sides by a tells us $a^0 = 1$.
This dictates the definition of the zero exponent!

$$a^0 := 1.$$

What about negative exponents?

$$a^{-b} \cdot a^b = a^{-b+b} = a^0 = 1$$

so dividing both sides by a^b gives $a^{-b} = \frac{1}{a^b}$.

So we must define

$$a^{-b} = \frac{1}{a^b}!$$

That is to say, the negative in the exponent indicates a reciprocal.

One could also write

$$\left(\frac{a}{b}\right)^{-c} = \left(\frac{b}{a}\right)^c.$$

Consider these examples:

1. $2^{-1} = \frac{1}{2}$
2. $-2^{-2} = -\frac{1}{4}$ (Note what the base is!)
3. $(-2)^{-2} = \frac{1}{4}$
4. $\left(\frac{2}{3}\right)^{-2} = \frac{9}{4}$
5. $2x^{-2} = \frac{2}{x^2}$
6. $\frac{3x}{y^{-3}z} = \frac{3xy^3}{z}$

1.3 Scientific Notation

Small and large numbers are difficult to write down (for example imagine writing down the distance in inches from here to the moon, or the width of a hair measured in miles). However, approximations can be easily written down in what is called *scientific notation.* We first note that, for example,

$$2.63 \cdot 10 = 263$$

and

$$\frac{63.4}{10} = 6.34.$$

Observe what happened to the decimal place when multiplying or dividing by 10.
Multiply the following:

$$3.45 \times 10^8 =$$

$$3.45 \times 10^{-5} = \frac{3.45}{10^5} =$$

The numbers on the left-hand side in the above two equalities are written in scientific notation, that is, each number has a specific format: each is a number between 1 and 10 including 1, multiplied by an integer power of 10 (positive or negative).

Here are a couple more examples of numbers written in scientific notation

$$2,450,000,000 = 2.45 \times 10^9.$$

$$0.0000067 = 6.7 \times 10^{-6}.$$

This notation makes it easy to multiply and divide such numbers:

$$\begin{aligned}
\frac{(3 \times 10^5)(2 \times 10^{-10})}{(1.2 \times 10^{-4})(2.0 \times 10^{-}6)} &= \left(\frac{3 \cdot 2}{1.2 \cdot 2}\right) \cdot \left(\frac{10^5 \cdot 10^{-10}}{10^{-4} \cdot 10^{-6}}\right) \\
&= \frac{6}{2.4} \cdot \frac{10^{-5}}{1} 10^{-10} \\
&= .25 \cdot 10^{-5} 10^{10} \\
&= .25 \cdot 10^5 \\
&= 2.5 \times 10^{-1} 10^5 \\
&= 2.5 \times 10^4
\end{aligned}$$

1.4 Rational Exponents

What about rational exponents?
$(3^{\frac{1}{2}})^2 = 3$ so $3^{\frac{1}{2}}$ is a solution to the equation $x^2 = 3$, but we know the positive solution is exactly $\sqrt{3}$. So $3^{\frac{1}{2}} := \sqrt{3}$.

Here are some other examples of rational exponents

$$(2^{\frac{1}{3}})^3 = 2^{\frac{1}{3}\frac{3}{1}} = 2, \text{ so, } 2^{\frac{1}{3}} \text{ is the solution to the equation } x^3 = 2. \text{ So, } 2^{\frac{1}{3}} = \sqrt[3]{2}$$

And,

$$(3^{\frac{1}{4}})^4 = 3^{\frac{1}{4}\cdot\frac{4}{1}} = 3, \text{ so, } 3^{\frac{1}{4}} \text{ is a solution to the equation } x^4 = 3, \text{ and taking it to be a positive sol}$$

More generally, $(a^{\frac{1}{b}})^b = a$ so $a^{\frac{1}{b}}$ is a solution to the equation $x^b = a$, which we know as $\sqrt[b]{a}$, so, taking the positive solution, we must define

$$a^{\frac{1}{b}} := \sqrt[b]{a}.$$

For example

1. $8^{1/3} = \sqrt[3]{8} = 2$
2. $\left(\frac{8}{27}\right)^{\frac{1}{3}} = \frac{2}{3}$
3. $4^{\frac{3}{2}} = (4^{\frac{1}{2}})^3 = (\sqrt{4})^3 = 2^3 = 8$
4. $8^{2/3} = (8^{1/3})^2 = 2^2 = 4$

Here are some examples that require us to use several rules we have learned in this chapter

1. $(9x^3y^{\frac{2}{3}})^{\frac{3}{2}} = 9^{\frac{3}{2}}x^{3\cdot\frac{3}{2}}y^{\frac{2}{3}\frac{3}{2}} = 27\sqrt{x}^9\,y$
2. $(xy^{-3})^{1/3} = x^{1/3}y^{-1} = \frac{\sqrt[3]{x}}{y}$
3. $xy^{-\frac{1}{2}}\left(\frac{x^{-2}}{y^{-\frac{2}{3}}x^2}\right)^{-\frac{1}{2}} = xy^{-\frac{1}{2}}\left(\frac{x}{y^{\frac{1}{3}}x^{-1}}\right) = \frac{x^2y^{-\frac{1}{2}}}{y^{\frac{1}{2}}x^{-1}} = \frac{x^2x}{y^{\frac{1}{2}}y^{\frac{1}{2}}} = \frac{x^3}{y}$