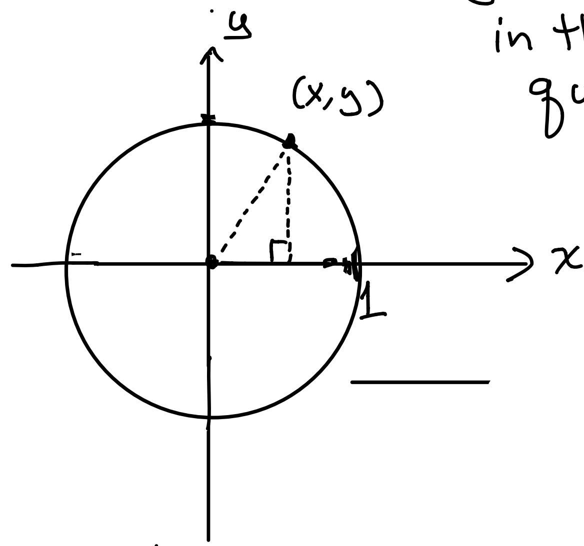
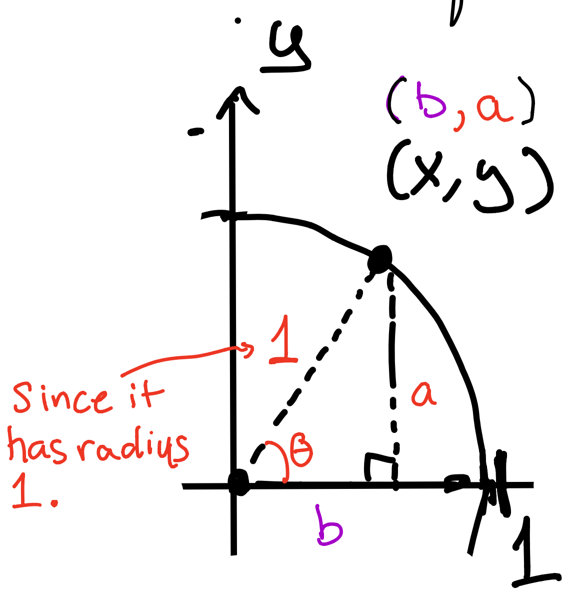
The unit circle and extending the meaning of the ratios.

Consider the circle with center $(0,0)$ and radius 1. Let (x,y) be a point in the 1st quadrant.



This is called the **unit circle**.

Form the right triangle as shown. Zooming in to that quadrant



$\frac{a}{1} = \sin\theta \Rightarrow a = \sin\theta$

So, $y = \sin\theta$.

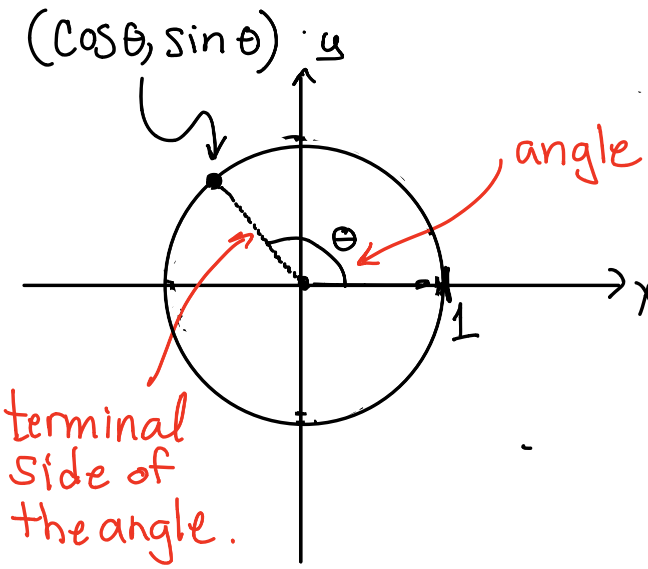
$\frac{b}{1} = \cos\theta \Rightarrow b = \cos\theta$

So, $x = \cos\theta$.

$\frac{a}{b} = \tan\theta$, so $\frac{y}{x} = \tan\theta$

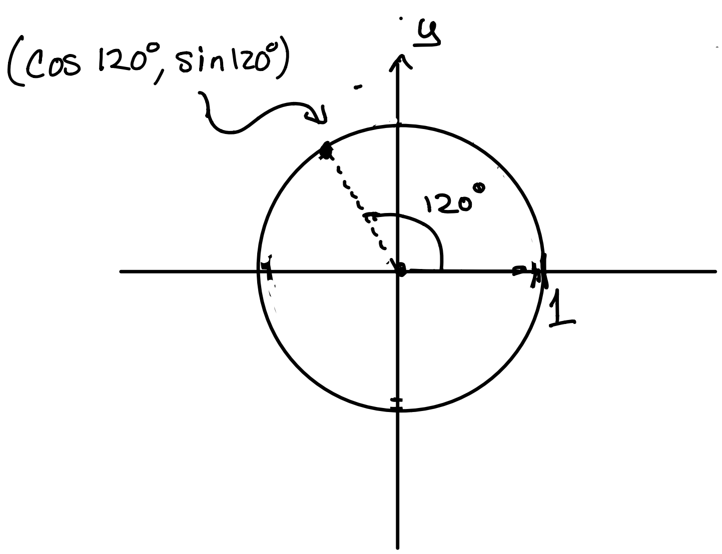
So $(x,y) = (\cos\theta, \sin\theta)$, for $0 < \theta < 90°$

We define $\cos\theta$ and $\sin\theta$ to be the x & y coordinates of the point where the terminal side of the angle with measure θ intersects with the unit circle.



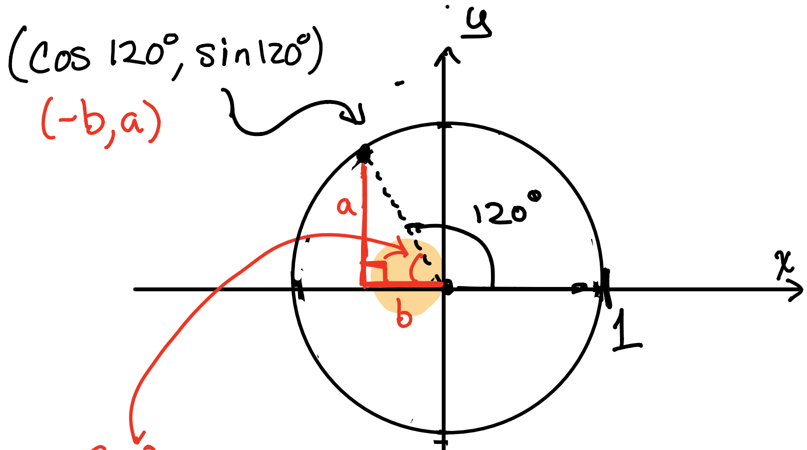
Examples:

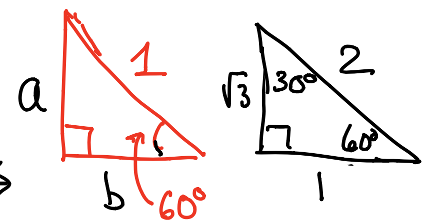
$\cos(120^\circ)$ is the x-coordinate of the point:



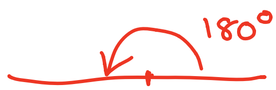
But what is its value?

We will relate it to a right triangle:





60° since



and $180° - 120° = 60°$

Comparing with our standard 30-60-90 triangle, we see $\frac{a}{1} = \frac{\sqrt{3}}{2}$ and $\frac{b}{1} = \frac{1}{2}$.

Putting this information in our picture tells us

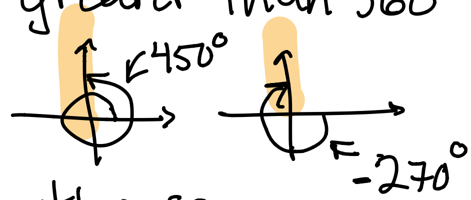
$$(\cos 120°, \sin 120°) = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right).$$

So, $\cos 120° = -\frac{1}{2}$.

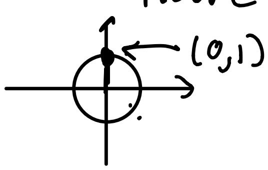
The triangle in red is called a reference triangle, and the angle marked in red is called the reference angle.

If the reference angle is not a special angle, we may use a calculator to estimate its value.

Also, angles may be greater than $360°$ or less than 0:

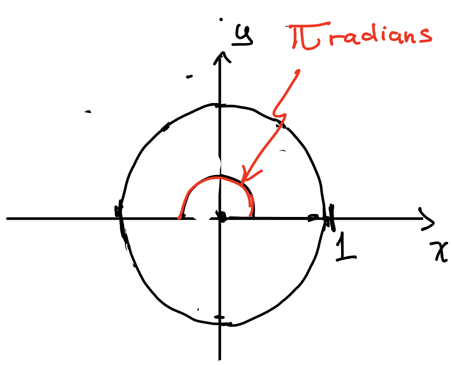


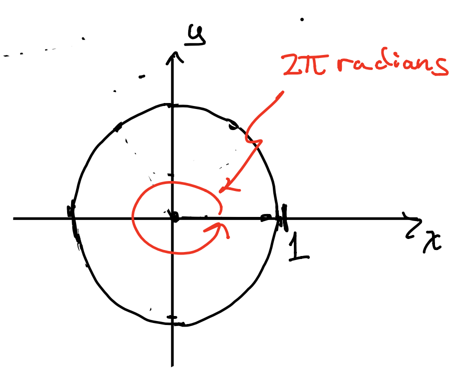
If the terminal side is the same for 2 angles the trigonometric ratios have the same value. For example,

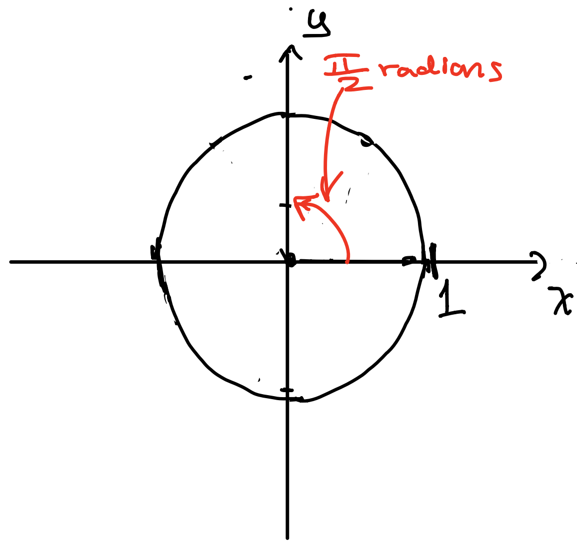


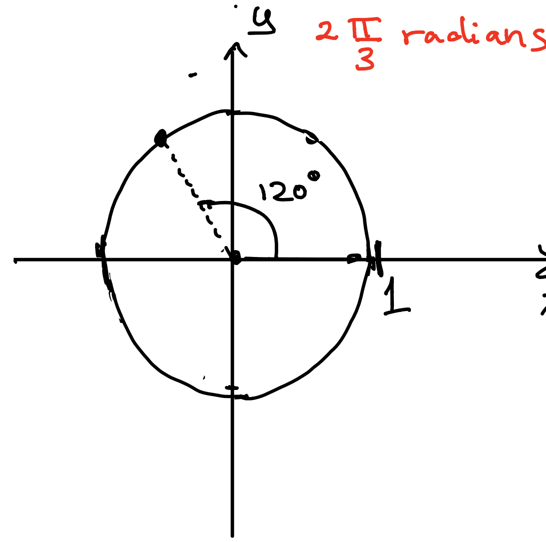
$$\sin(-270°) = \sin(90°) = 1.$$

It is very convenient in Calculus to use a measure of angle called radians that is not degrees. It is defined as the arclength of the arc corresponding to the angle. So, (noting Circumference of the unit ⊙ is 2π) for instance









In general $\theta^\circ = \frac{\theta}{180^\circ}\pi$ radians

and $X \text{ radians} = \frac{X}{\pi} 180^\circ$.

Examples

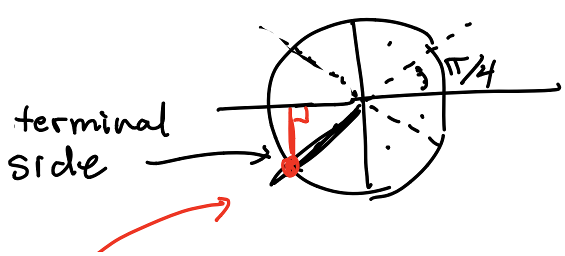
$$630^\circ = \frac{630^\circ}{180^\circ}\pi = \frac{21}{6}\pi$$

and

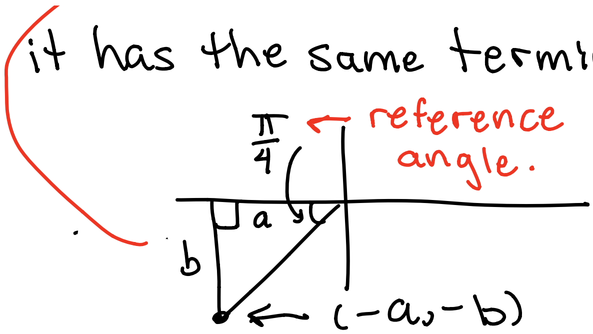
$$\frac{-7\pi}{3} = -\frac{7\pi}{3\cdot\pi}180^\circ = -7.60^\circ$$

$$= -420^\circ.$$

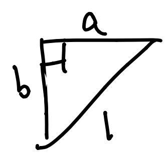
Example: $\tan\left(-\frac{11\pi}{4}\right) =$

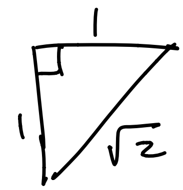


$-\frac{11\pi}{4}$ is coterminal with $-\frac{3\pi}{4}$, i.e., it has the same terminal side.



$$\boxed{\text{Note } -\frac{11\pi}{4} = -\frac{11\pi}{4\pi}\,180^\circ = -11 \cdot 45^\circ}$$

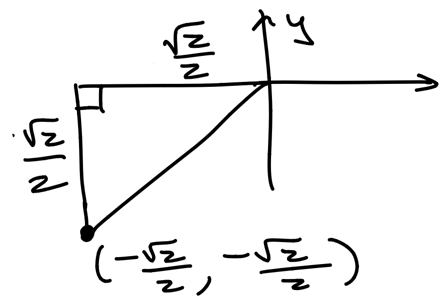




$$\frac{a}{1} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\& \quad \frac{b}{1} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}.$$

So

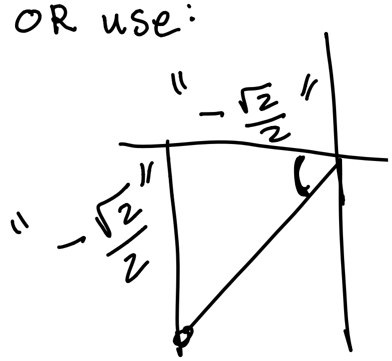


So $\cos\left(-\frac{11\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

and $\sin\left(\frac{-11\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

and $\tan\left(-\frac{11\pi}{4}\right) = \frac{-\frac{\sqrt{2}}{2}}{-\frac{\sqrt{2}}{2}} = 1.$

OR use:



For example:

$$\cos\left(\frac{-11\pi}{4}\right) = \frac{\text{signed adj}}{\text{hyp}} = \frac{-\frac{\sqrt{2}}{2}}{1} = \frac{-\sqrt{2}}{2}$$