Exponents 1: (Integer Exponents).

We know that $3 \cdot 4 = 4+4+4$

$\uparrow$ counts the number of 4's being added!

Similarly $4^3 = 4 \cdot 4 \cdot 4$

counts the number of 4's being multiplied.

In the expression a^b, a is called the base and b is called the exponent.

a^b is called an exponential expression.

b is the number of a's being multiplied.

Properties:

① $a^b a^c = a^{b+c}$:

Example of why: $2^3 \cdot 2^4 = (2 \cdot 2 \cdot 2) \cdot (2 \cdot 2 \cdot 2 \cdot 2)$

$= 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$

$= 2^7$.

Since the exponent counts the number of 2's being multiplied.

② $(a^b)^c = a^{b \cdot c}$:

Example of why: $(4^2)^3 = 4^2 \cdot 4^2 \cdot 4^2$

$= (4 \cdot 4) \cdot (4 \cdot 4) \cdot (4 \cdot 4)$

$= 4^6 \leftarrow$ 3 pairs of 4's.

(3a) $a^c b^c = (a \cdot b)^c$:

Example of why: $2^3 5^3 = (2 \cdot 2 \cdot 2)(5 \cdot 5 \cdot 5)$

$= (2 \cdot 5)(2 \cdot 5)(2 \cdot 5) = (2 \cdot 5)^3$.

(3b) $\dfrac{a^c}{b^c} = \left(\dfrac{a}{b}\right)^c$

Example of why: $\dfrac{5^3}{4^3} = \dfrac{5\cdot5\cdot5}{4\cdot4\cdot4} = \dfrac{5}{4}\cdot\dfrac{5}{4}\cdot\dfrac{5}{4}$

$= \left(\dfrac{5}{4}\right)^3.$

Note: Exponents $\xrightarrow{\text{distributes over}}$ Multiplication & division $\xrightarrow{\text{distributes over}}$ Addition & Subtraction.

Example: $\left(3\cdot x^5\cdot y^3\right)^2 \left(2x^6y\right)^3$

(3a) $\longrightarrow$ $= 3^2 (x^5)^2 (y^3)^2 \cdot 2^3 (x^6)^3 y^3$

(2) $\longrightarrow$ $= 9\cdot 8\cdot x^{10}\cdot y^6\cdot x^{18} y^3$

(1) $\longrightarrow$ $= 72\cdot x^{28} y^9.$

Try: Simplify:

$\left(-2x^5y^2\right)^3 \left(3x^6y^5\right)^2$

Example: $\dfrac{(3x^5y)^2}{(9xy^2)^2} \overset{(3a)}{=} \dfrac{3^2\cdot x^{10} y^2}{9^2 x^2 y^4} = \dfrac{x^8}{9y^2}.$ $2\cdot 2$

(Here, $\dfrac{x^{10}}{x^2} = \dfrac{\overbrace{x\cdot x\cdot x\cdots x}^{10\ xs}}{x\cdot x} = x^8$; $\dfrac{y^2}{y^4} = \dfrac{y\cdot y}{y\cdot y\cdot y\cdot y} = \dfrac{1}{y^2}$)

Try: What happens if you first use property (3b)?

So far 3^0, for example has no meaning.
But if we want our properties to continue to work,

$$3^0 \cdot 3^1 = 3^{0+1} = 3^1.$$ So, 3^0, whatever it is,

should satisfy $3^0 \cdot 3 = 3$ so dividing by 3,
we see 3^0 must be 1. 3 could be replaced with any non-zero number and so,

(I) $$a^0 := 1 \quad \text{for } a \neq 0.$$

But 2^{-3} has no meaning so far.
If our properties for exponents also hold for negative exponents,

$$2^{-3} \cdot 2^3 = 2^{-3+3} = 2^0 = 1$$

So, whatever 2^{-3} is, it satisfies

$$2^{-3} \cdot 2^3 = 1$$

$$\Rightarrow 2^{-3} = \frac{1}{2^3}.$$

So we have

(II) $$a^{-b} := \frac{1}{a^b}.$$

Ans:
$-72x^{27}y^{16}$

So, the negative exponent means

"replace the base with it's reciprical".

Example: $\left(\frac{2}{3}\right)^{-3} = \left(\frac{3}{2}\right)^{3} = \frac{3^3}{2^3} = \frac{27}{8}$

Example: $\frac{x^{-3}y}{x^2y^{-2}} = \frac{y}{x^2}\cdot x^{-3}\cdot\frac{1}{y^{-2}}$

$$= \frac{y}{x^2}\cdot\frac{1}{x^3}\cdot y^2$$

$$\Rightarrow \frac{x^{-3}y}{x^2y^{-2}} = \frac{y\cdot y^2}{x^2x^3} = \frac{y^3}{x^5}$$

Example: $3\left(\frac{2x^{-2}y^3z^{-5}}{3xy^2z^{-2}}\right)^2\left(\frac{-x^2y^{-2}}{zy}\right)^{-3}$

$$= 3\left(\frac{2y^3\cdot z^2}{3xy^2\cdot x^2z^5}\right)^2\left(\frac{-x^2}{zy\cdot y^2}\right)^{-3}$$

$$= 3\left(\frac{2y}{3x^3z^5}\right)^2\left(\frac{-zy^3}{x^2}\right)^3$$

$$= 3\cdot\frac{2^2\cdot y^2}{3^2\cdot x^6z^{10}}\cdot\frac{(-1)^3z^3y^9}{x^6}$$

$$= \frac{3\cdot 2^2\cdot(-1)^3y^2y^9z^3}{3^2\cdot x^6x^6z^{10}}$$

$$= -\frac{4y^{11}}{3x^{12}z^7}.$$

Ans

$$\frac{(3x^5y)^2}{(9xy^2)^2} = \left(\frac{3x^5y}{9xy^2}\right)^2 = \left(\frac{x^4}{3y}\right)^2 = \frac{x^8}{3^2y^2} = \frac{x^8}{9y^2}$$

Try: Simplify and write your answer using positive exponents.

$$\frac{3x}{y^{-2}}\left(\frac{x^2y^{-5}}{z^{-2}x^{-2}}\right)^{-2}\left(\frac{3xy^{-1}}{4z^2y^{-3}x^2}\right)^2$$

Try: What happens in the above worked example if you use property 3 a,b first?

Note: With definitions I & II, properties of exponents work for positive and negative integers.

Ans.

$$\text{Expression} = 3xy^2 \cdot \frac{y^{10}}{x^8z^4} \cdot \frac{3^2y^4}{4^2x^2z^4} = \frac{27\,y^{16}}{16\,x^9z^8}$$