11/10/2021 Session 16:

Applications: Half-life + compound interest

Def: Let f be an exponential function $f(x) = c \cdot b^x$ (domain: $\mathbb{R}$). Then we say $f(x)$ has a half-life of h, if the base is given by

$$b = \left(\tfrac{1}{2}\right)^{1/h}.$$

$$\boxed{f(x) = c \cdot \left(\tfrac{1}{2}\right)^{\frac{x}{h}}}$$

Let x = time

Note: $f(x+h) = \frac{1}{2} f(x)$

Ex An isotope decays within 20 hours from 5g to 2.17g. Find the half-life.

$$\frac{2.17}{5} = \frac{5 \cdot \left(\tfrac{1}{2}\right)^{\frac{20}{h}}}{5}$$ ← solve for h!

$$\frac{2.17}{5} = \left(\frac{1}{2}\right)^{20/h}$$

$$\ln\left(\frac{2.17}{5}\right) = \frac{20}{h}\ln\left(\frac{1}{2}\right)$$

$$h \cdot \ln\left(\frac{2.17}{5}\right) = \frac{20}{\cancel{h}} \ln\left(\frac{1}{2}\right) \cdot \cancel{h}$$

$$\frac{h \cancel{\ln\left(\frac{2.17}{5}\right)}}{\cancel{\ln\left(\frac{2.17}{5}\right)}} = \frac{20\ln\left(\frac{1}{2}\right)}{\ln\left(2.17/5\right)}$$

$$\boxed{h = 20\,\frac{\ln(1/2)}{\ln(2.17/5)} \approx 16.6 \text{ hrs}}$$

An important isotope is the radioisotope carbon-14. It decays with a half-life of 5730 years! We can use this knowledge to "carbon date" organic materials.

Ex A dead animal at an archeological site has lost 41.3% of its carbon-14. When did the animal die?

$$f(x) = \underset{\substack{\uparrow \\ \text{initial amount}}}{C} \cdot \left(\frac{1}{2}\right)^{\frac{x}{\underset{\substack{\uparrow \\ \text{half life}}}{h}}}$$

$$(1-.413)\cdot \cancel{C} = \cancel{C}\cdot\left(\tfrac{1}{2}\right)^{\frac{x}{5730}}$$ Solve for x!

amount remaining ↑ (.413) — initial amt ↑ (C) — ↑ ½ life of carbon (5730)

$$0.587=\left(\tfrac{1}{2}\right)^{\frac{x}{5730}}$$

$$\ln(0.587)=\ln\left(\tfrac{1}{2}\right)^{\frac{x}{5730}}$$

$$\frac{\ln(0.587)}{\ln(1/2)}=\frac{x}{5730}\cdot\frac{\ln(1/2)}{\ln(1/2)}$$

$$\cancel{5730}\cdot\frac{x}{\cancel{5730}}=\frac{\ln(0.587)}{\ln(1/2)}\cdot 5730$$

$$x=5730\cdot\frac{\ln(0.587)}{\ln(1/2)}\approx \boxed{4403.9 \text{ years!}}$$