10/27/2021

From last class:

Exponential Functions:

$$y = c \cdot b^x$$

b ← base

c ↑ constant #

Their inverses:

Logarithmic Functions:

$$\log_b(x) = y$$

Ex $\log_{10}(100) = 2$ "The log is the exponent!"

base ↑ exponent ↑

⇕

$$10^2 = 100$$

Lesson 14: **Exponential & Logarithmic Equations**

Recall some properties of exponents!

1) $b^{x+y} = b^x \cdot b^y$

2) $b^{x-y} = \dfrac{b^x}{b^y}$

3) $(b^m)^n = b^{m \cdot n}$

Properties of logarithms:

$b>0$
$b \neq 1$
$x, y > 0$

$$\boxed{\begin{aligned} &1)\ \log_b(x \cdot y) = \log_b(x) + \log_b(y) \\ &2)\ \log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y) \\ &3)\ \log_b(x^n) = n \cdot \log_b(x) \end{aligned}}$$

Ex Combine the terms using the properties of logarithms so as to write one logarithm

a) $\frac{1}{2}\ln(x) - \ln(y)$
(use property 3)

"ln" = natural logarithm
$\log_e$ $e \approx 2.718\ldots$

$= \ln(x^{1/2}) - \ln(y)$ (use property 2)

$= \ln\left(\frac{x^{1/2}}{y}\right) = \boxed{\ln\left(\frac{\sqrt{x}}{y}\right)}$

b) $2\ln(x) - \frac{1}{3}\ln(y) - \frac{7}{5}\ln(z)$

(use property 3 first)

$$= \underbrace{\ln(x^2) - \ln(y^{1/3})} - \ln(z^{7/5})$$

work our way from left to right & use property 2

$$= \ln\left(\frac{x^2}{y^{1/3}}\right) - \ln(z^{7/5})$$

$$= \ln\left(\left(\frac{x^2}{y^{1/3}}\right) / z^{7/5}\right)$$

$$\boxed{= \ln\left(\frac{x^2}{y^{1/3} \cdot z^{7/5}}\right)}$$

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c) $5 + \log_2(a^2 - b^2) - \log_2(a-b)$

Rewrite 5 as a logarithm: $\boxed{\log_2(2^5) = 5}$

$$= \log_2(2^5) + \log_2(a^2 - b^2) - \log_2(a-b)$$

$$= \underbrace{\log_2(32) + \log_2(a^2 - b^2)} - \log_2(a-b)$$

Work left to right use property 1

$$= \log_2(32(a^2 - b^2)) - \log_2(a-b)$$

use property 2

$$= \log_2 \left(\frac{32(a^2-b^2)}{a-b} \right)$$

Recall:
$a^2-b^2=(a+b)(a-b)$

$$= \log_2 \left(\frac{32(a+b)\cancel{(a-b)}}{\cancel{(a-b)}} \right)$$

$$\boxed{= \log_2 (32(a+b))}$$

Going backwards:

Ex Write the expression in terms of elementary logarithms $u=\log_b(x)$, $v=\log_b(y)$, $w=\log_b(z)$. $(x, y, z>0)$

a) $\log\left(\sqrt{\sqrt{x}\cdot y^3}\right)$

$$= \log\left(\left(x^{1/2}\cdot y^3\right)^{1/2}\right)$$

(use property 3)

$$= \frac{1}{2}\log\left(x^{1/2}\cdot y^3\right)$$

use property 1

$$= \frac{1}{2}\left[\log\left(x^{1/2}\right)+\log\left(y^3\right)\right]$$

$$= \frac{1}{2}\left[\frac{1}{2}\underbrace{\log(x)} + 3\underbrace{\log(y)}\right]$$

$$\boxed{= \frac{1}{2}\left[\frac{1}{2}u + 3v\right]}$$

Observations:

(1) $b^x = b^y \iff x = y$

(2) $\log_b(x) = \log_b(y) \iff x = y$

Ex Solve for x

a) $10^{2x-8} = 0.01$

use (1), first rewrite the right hand side of the equation in base 10.

Facts:

$10^{-2} = \frac{1}{10^2} = \frac{1}{100} = 0.01$

$$10^{2x-8} = 10^{-2}$$

$$\Rightarrow 2x - 8 = -2$$

Just a linear equation! Solve it!

$\boxed{x = 3}$ Check it! $10^{2x-8} = 0.01$

$$10^{2(3)-8} \stackrel{?}{=} 0.01$$

$$10^{-2} = 0.01 \checkmark$$

b) $5^{3x+1} = 25^{4x-7}$ $\boxed{\text{Know: } 5^2 = 25}$

$5^{3x+1} = (5^2)^{4x-7}$

$5^{3x+1} = 5^{2(4x-7)} \Rightarrow 3x+1 = 2(4x-7)$

Now solve for x

$\boxed{x = 3}$

(Check in original equation)

c) $\ln(3x-5) = \ln(x-1)$

$\Rightarrow 3x-5 = x-1$ Solve!

$\boxed{x = 2}$

Check in original: $\ln(3\cdot 2-5) \stackrel{?}{=} \ln(2-1)$

$\ln(1) \stackrel{\checkmark}{=} \ln(1)$

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d) $\underbrace{\log_6(x) + \log_6(x+4)} = \log_6(5)$

use property 1 of logs to rewrite the left hand side of the equation as a single logarithm

$\log_6(x(x+4)) = \log_6(5)$

$x(x+4) = 5$

$x^2 + 4x - 5 = 0$ Solve the quadratic equation

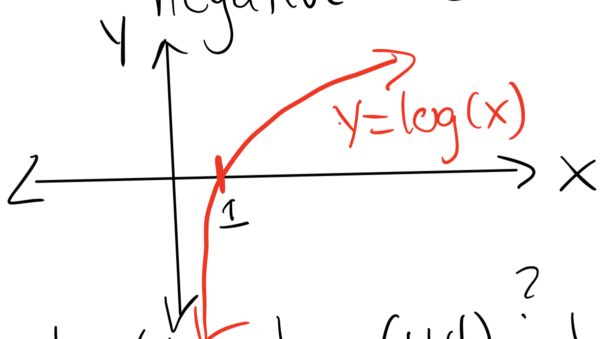
$(x+5)(x-1) = 0$

reject $\boxed{\cancel{x=-5}}$ $\boxed{x=1}$ Keep

We must check both potential solutions in the original equation: $\log_6(x)+\log_6(x+4)=\log_6(5)$

Check $\cancel{x=-5}$: $\log_6(-5)+\log_6(-5+4)=\log_6(5)$

$\log_6(x)$ is not defined for negative #'s



Check $x=1$: $\log_6(1)+\log_6(1+4)\stackrel{?}{=}\log_6(5)$

$\cancel{\log_6(1)}^{0}+\log_6(5)\stackrel{?}{=}\log_6(5)$

$\log_6(1\cdot 5)=\log_6(5)$

You try:

e) $\log_3(x-2)+\log_3(x+6)=2$

rewrite as a log

$\boxed{\log_3 3^2=2}$

$$\log_3\left((x-2)\cdot(x+6)\right) = \log_3(3^2)$$

$(x-2)(x+6) = 9$ Quadratic Equation

$$x^2 \underbrace{-2x+6x}-12 = 9$$
$$-9 \quad -9$$

$$x^2+4x-21=0$$

$$(x+7)(x-3)=0$$

$\boxed{\cancel{x=-7}}$ reject $\boxed{x=3}$ Keep Check in original equation!

$$\log_3(x-2)+\log_3(x+6)=2$$

$\log_3(-7-2)$

-9 $\log_3(-9)$ undefined!

Ex Solve

a) $3^{x+5}=8$

Notice right away that we cannot rewrite so that each side has the same base! What we can do is take the logarithm of both sides. Use log or ln.

$$\log(3^{x+5}) = \log(8)$$

use property 3 of logs

$$(x+5)\log(3) = \log(8)$$

Note: log(3) + log(8) are just numerical values.

Solve for x

$$x\log(3) + 5\log(3) = \log(8)$$

$$-5\log(3) \qquad -5\log(3)$$

$$\frac{x\cancel{\log(3)}}{\cancel{\log(3)}} = \frac{\log(8) - 5\log(3)}{\log(3)}$$

$$x = \frac{\log(8) - 5\log(3)}{\log(3)} \approx -3.107$$

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b) $5^{x-7} = 2^x$

$$\ln(5^{x-7}) = \ln(2^x)$$ (use property 3 of logs)

$$(x-7)\ln(5) = x\ln(2)$$

$$x\ln(5) - 7\ln(5) = x\ln(2)$$

$$+7\ln(5) \qquad +7\ln(5)$$

$$-x\ln(2) \qquad -x\ln(2)$$

$$x\ln(5) - x\ln(2) = 7\ln(5)$$

$$\frac{x\,\cancel{(\ln(5) - \ln(2))}}{\cancel{\ln(5) - \ln(2)}} = \frac{7\ln(5)}{\ln(5) - \ln(2)}$$

$$x = \frac{7\ln(5)}{\ln(5) - \ln(2)} \approx \boxed{12.3}$$