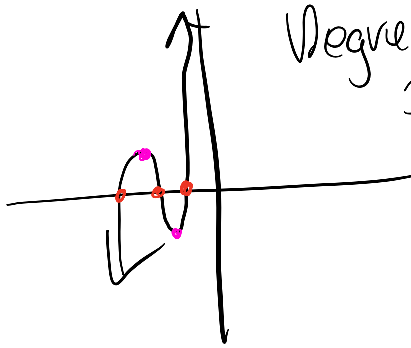
10/13/2021

Consider $f(x)$ with roots x_1, x_2 and x_3



Degree 3
3 roots
2 extrema

Then $f(x)$ has factors $(x-x_1)(x-x_2)(x-x_3)$.

A polynomial of degree n has at most $\underline{n}$ roots, and at most $\underline{n-1}$ extreme values.

Roots of polynomials

Rational Roots Theorem: Consider the equation

$$(**)\ a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = 0$$

where every coefficient a_i is an integer and $a_n \neq 0 \neq a_0$. Assume that $x = \frac{p}{q}$ is a solution of $(**)$ and the fraction $\frac{p}{q}$ is completely reduced. Then a_0 is an integer multiple of p and a_n is an integer multiple of q. Therefore all possible rational solutions of $(**)$ are fractions $x = \frac{p}{q}$ where p is a factor of a_0 and q is a factor of a_n.

Ex Find all the real roots of

$$f(x) = \boxed{2}x^3 + 11x^2 - 2x \boxed{-2}$$

1) List factors: $\pm 1, \pm 2$

2) List of fractions: $\pm\frac{1}{1}, \pm\frac{2}{1}, \pm\frac{1}{2}$ $\boxed{= \pm 1, \pm 2, \pm\frac{1}{2}}$

3) Check our list of potential roots to see if any are roots i.e. $f(x_i) = 0$

$f(1), f(-1), f(2), f(-2), f(-\frac{1}{2}), \boxed{f(\frac{1}{2})}$ ✓

$f(x) = 2x^3 + 11x^2 - 2x - 2$ has root $x = \frac{1}{2}$

$\Rightarrow (x - \frac{1}{2})$ is a factor

$$f(x) = 2x^3 + 11x^2 - 2x - 2 = (x - \tfrac{1}{2})(\ ??\ )$$

$$\begin{array}{r}
2x^2 + 12x + 4 \\
x - \tfrac{1}{2}\ \overline{)\ 2x^3 + 11x^2 - 2x - 2} \\
-(2x^3 - x^2) \qquad\qquad \\
\hline
12x^2 - 2x \qquad \\
-(12x^2 - 6x) \qquad \\
\hline
4x - 2 \\
-(4x - 2) \\
\hline
0
\end{array}$$

$$f(x)=\left(x-\tfrac{1}{2}\right)(2x^2+12x+4)=2\underbrace{\left(x-\tfrac{1}{2}\right)}_{\text{prime}}\underbrace{(x^2+6x+2)}_{\text{needs factoring!}}$$

Can't factor → use quadratic formula!

$$x^2+6x+2=0$$

$$\boxed{x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}}$$

$a=1 \quad b=6 \quad c=2$

$$x=-3\pm\sqrt{7}$$

$$x=\frac{-6\pm\sqrt{6^2-4(1)(2)}}{2(1)}$$

$$x=\frac{-6\pm\sqrt{36-8}}{2}=\frac{-6\pm\sqrt{28}}{2}$$

$$x=\frac{-6\pm\sqrt{4\cdot 7}}{2}=\frac{-6\pm\sqrt{4}\cdot\sqrt{7}}{2}$$

$$x=\frac{\overset{-3}{\cancel{-6}}\pm\cancel{2}\sqrt{7}}{\cancel{2}}=\boxed{-3\pm\sqrt{7}}$$

We have $-3+\sqrt{7}$ is a root so $(x-(-3+\sqrt{7}))$ is a factor.

$-3-\sqrt{7}$ is root so $(x-(-3-\sqrt{7}))$ is a factor.

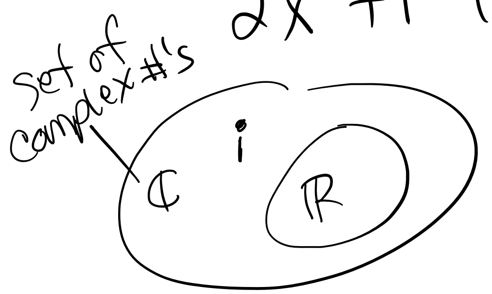
$$f(x)=2x^3+11x^2-2x-2=\boxed{2\left(x-\tfrac{1}{2}\right)(x-(-3+\sqrt{7}))(x-(-3-\sqrt{7}))}$$

Fundamental Theorem of Algebra:

Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ be a non-constant polynomial. Then there exists a complex # c which is a root of f.

Ex 4 is a constant poly

$2x^2+1$ is a non-constant polynomial.



Notes: 1) The coefficients don't need to be real!

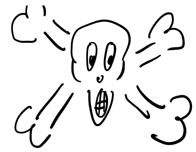
2) There may or may not be any real roots.

Ex $x^2+1=0$

$x^2=-1$

$x = \pm\sqrt{-1} = \pm i \leftarrow$ imaginary.

$+i$ and $-i$ are called complex conjugates.

Ex $f(x)=x^3+1$  "The Shining" "Insidious"

Factor This!!

The rational roots theorem says ± 1 may be roots.

Checked $x=-1$ is a root $\Rightarrow (x-(-1)) = (x+1)$ is a factor

$$f(x)=x^3+1=(x+1)(\;??\;)$$

$$\begin{array}{r} x^2-x+1 \\ x+1 \overline{\smash{)}\,x^3+0x^2+0x+1} \\ \vdots \\ \overline{\quad 0} \end{array}$$

$$f(x)=x^3+1=\underbrace{(x+1)}_{\text{prime}}\underbrace{(x^2-x+1)}_{\text{can be factored}}$$

use the quadratic formula

$$x^2-x+1=0 \qquad a=1 \quad b=-1 \quad c=1$$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a} \qquad x=\frac{-(-1)\pm\sqrt{(-1)^2-4(1)(1)}}{2(1)}=\frac{1\pm\sqrt{-3}}{2}$$

$X = \frac{1 \pm i\sqrt{3}}{2} \Rightarrow X = \frac{1}{2} + i\frac{\sqrt{3}}{2} \quad X = \frac{1}{2} - i\frac{\sqrt{3}}{2}$ roots

so

$\left(X - \left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)\right)$ and $\left(X - \left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)\right)$

are factors

$$\boxed{X^3 + 1 = (X+1)\left(X - \left(\tfrac{1}{2} + i\tfrac{\sqrt{3}}{2}\right)\right)\left(X - \left(\tfrac{1}{2} - i\tfrac{\sqrt{3}}{2}\right)\right)}$$

and complex conjugates!

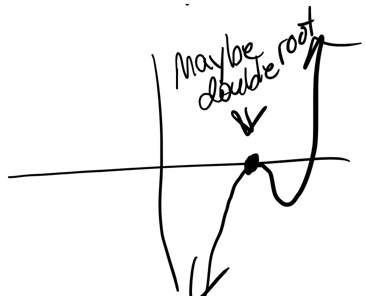
Observations:

1) Every polynomial of degree n can be factored $f(X) = m(X - c_1)(X - c_2) \cdots (X - c_n)$

2) Every polynomial of degree n has at most n root. These roots may be real or complex.

3) The factor $(X - c)$ for the root c could appear multiple times. If $(X - c)^K$ Then K is called the multiplicity of c.

Ex $f(X) = (X-2)^2(X-1)$ $\quad X = 2$ is a root of multiplicity 2.



4) If $f(x)$ has only real coefficients a_i and $c = a + bi$ is a complex root of f Then The complex conjugate $\bar{c} = a - bi$ is also a root of f.