Handout for section 9.3

Equivalent Vectors

Every vector $\overrightarrow{PQ}$ is equivalent to a vector $\overrightarrow{OR}$ with initial point at the origin:
If $P = (x_1, y_1)$ and $Q = (x_2, y_2)$, then

$$\overrightarrow{PQ} = \overrightarrow{OR}, \qquad \text{where } R = (x_2 - x_1, y_2 - y_1).$$

Scalar Multiplication

If k is a real number and $\mathbf{v} = \langle a, b\rangle$ is a vector, then

$$k\mathbf{v} \text{ is the vector } \langle ka, kb\rangle.$$

The vector $k\mathbf{v}$ is called a **scalar multiple** of $\mathbf{v}$.

Geometric Interpretation of Scalar Multiplication

The *magnitude* of the vector $k\mathbf{v}$ is $|k|$ times the length of $\mathbf{v}$, that is,

$$\|k\mathbf{v}\| = |k| \cdot \|\mathbf{v}\|.$$

The *direction of* $k\mathbf{v}$ is the same as that of $\mathbf{v}$ when k is positive and opposite that of $\mathbf{v}$ when k is negative.

Vector Addition

If $\mathbf{u} = \langle a, b\rangle$ and $\mathbf{v} = \langle c, d\rangle$, then

$$\mathbf{u} + \mathbf{v} = \langle a + c, b + d\rangle.$$

Geometric Interpretations of Vector Addition

1. If $\mathbf{u}$ and $\mathbf{v}$ are vectors with the same initial point P, then $\mathbf{u} + \mathbf{v}$ is the vector $\overrightarrow{PQ}$, where $\overrightarrow{PQ}$ is the diagonal of the parallelogram with adjacent sides $\mathbf{u}$ and $\mathbf{v}$.
2. If the vector $\mathbf{v}$ is moved (without changing its magnitude or direction) so that its initial point lies on the endpoint of the vector $\mathbf{u}$, then $\mathbf{u} + \mathbf{v}$ is the vector with the same initial point P as $\mathbf{u}$ and the same terminal point Q as $\mathbf{v}$.

Vector Subtraction

If $\mathbf{u} = \langle a, b\rangle$ and $\mathbf{v} = \langle c, d\rangle$, then $\mathbf{u} - \mathbf{v}$ is the vector

$$\begin{aligned}\mathbf{u} + (-\mathbf{v}) &= \langle a, b\rangle + \langle -c, -d\rangle \\ &= \langle a - c, b - d\rangle.\end{aligned}$$

Properties of Vector Addition and Scalar Multiplication

For any vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ and any scalars r and s,

1. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
3. $\mathbf{v} + \mathbf{0} = \mathbf{v} = \mathbf{0} + \mathbf{v}$
4. $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$
5. $r(\mathbf{u} + \mathbf{v}) = r\mathbf{u} + r\mathbf{v}$
6. $(r + s)\mathbf{v} = r\mathbf{v} + s\mathbf{v}$
7. $(rs)\mathbf{v} = r(s\mathbf{v}) = s(r\mathbf{v})$
8. $1\mathbf{v} = \mathbf{v}$
9. $0\mathbf{v} = \mathbf{0}$ and $r\mathbf{0} = \mathbf{0}$

Unit Vectors

If $\mathbf{v}$ is a nonzero vector, then $\frac{1}{\|\mathbf{v}\|}\mathbf{v}$ is a unit vector with the same direction as $\mathbf{v}$.

Components of the Direction Angle

If $\mathbf{v} = \langle a, b\rangle = a\mathbf{i} + b\mathbf{j}$, then

$$a = \|\mathbf{v}\| \cos\theta \qquad \text{and} \qquad b = \|\mathbf{v}\| \sin\theta$$

where θ is the direction angle of $\mathbf{v}$.