10/13/2021

Roots of polynomials

Rational Roots Theorem: Consider the equation

(Polynomial Equation) $a_n x^n + a_{n-1} x^{n-1} + \quad + a_1 x + a_0 = 0$ **

where every coefficient is an integer $a_0 \neq 0 \neq a_n$. Assume that $x = \frac{p}{q}$ is a solution of ** and $\frac{p}{q}$ is completely reduced. Then a_0 is an integer multiple of p and a_n is an integer multiple of q.

i.e. p is a factor of a_0 (constant)
q is a factor of a_n (leading coefficient)

Strategy: Given a polynomial equation
1) Factor leading coeff (q's) + constant term (p's)
2) Form quotients of the factors $\frac{p\text{'s}}{q\text{'s}}$
3) Check the list for roots!!

Caveat: This give real roots only!

Ex Given $f(x) = 2x^3 + 11x^2 - 2x - 2 = 0$

1) List of factors : $\boxed{\pm 1, \pm 2}$

2) List of quotients : $\boxed{\pm\frac{1}{1}, \pm\frac{2}{1}, \pm\frac{1}{2}}$

: $\boxed{\pm 1, \pm 2, \pm\frac{1}{2}}$

This list is a list of potential 0 roots for $2x^3 + 11x^2 - 2x - 2$

3) Check! $\cancel{f(1)}, \cancel{f(-1)}, \cancel{f(2)}, \cancel{f(-2)}, \boxed{f(\tfrac{1}{2})}, \cancel{f(-\tfrac{1}{2})}$

use desmos! $x = \frac{1}{2}$ is a root

$\Rightarrow (x - \frac{1}{2})$ is a factor!

So far $f(x) = 2x^3 + 11x^2 - 2x - 2 = 0$

$= (x - \frac{1}{2})(2x^2 + 12x + 4) = 0$

$$
\begin{array}{r|l}
 & 2x^2 + 12x + 4 \\
\hline
x - \frac{1}{2} & 2x^3 + 11x^2 - 2x - 2 \\
 & -(2x^3 - x^2) \\
\hline
 & 12x^2 - 2x \\
 & -(12x^2 - 6x) \\
\hline
 & 4x - 2 \\
 & -(4x - 2) \\
\hline
 & 0
\end{array}
$$

$$f(x) = 2\underbrace{\left(x - \tfrac{1}{2}\right)}_{}\overbrace{}^{\text{prime}}\left(x^2+6x+2\right) = 0$$

(prime) (factor more)

(factored out 2)

Use the quadratic formula!

$$x^2+6x+2=0 \qquad x = \frac{-b \pm \sqrt{b^2-4ac}}{2(a)}$$

$\boxed{a=1}$ $\boxed{b=6}$ $\boxed{c=2}$

$$x = \frac{-6 \pm \sqrt{6^2-4(1)(2)}}{2 \cdot 1} = \frac{-6 \pm \sqrt{36-8}}{2} = \frac{-6 \pm \sqrt{28}}{2}$$

$$= \frac{-6 \pm \sqrt{4 \cdot 7}}{2} = \frac{\overset{-3}{\cancel{-6}} \pm \cancel{2}\sqrt{7}}{\cancel{2}} = \boxed{-3 \pm \sqrt{7}}$$

Remember if we have a root c then $x-c$ is a factor

$\Rightarrow$ $-3+\sqrt{7}$ is a root then $[x-(-3+\sqrt{7})]$ is a factor!

$-3-\sqrt{7}$ is a root then $[x-(-3-\sqrt{7})]$ is a factor!

$$\boxed{f(x) = 2\left(x-\tfrac{1}{2}\right)\left(x-(-3-\sqrt{7})\right)\left(x-(-3+\sqrt{7})\right)=0}$$

We know the roots: $\boxed{-\tfrac{1}{2}, -3+\sqrt{7}, -3-\sqrt{7}}$

The Fundamental Theorem of Algebra:

Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ be a non-constant polynomial. Then there exists a complex # c which is a root of f.

Notes:

a) a_i (coefficients) can be complex #'s

b) In general there may not be a real root

$$f(x) = x^2+1 = 0$$

$$\rightarrow \quad x^2 = -1$$

$$x = \pm\sqrt{-1} = \pm i \leftarrow \text{imaginary}$$

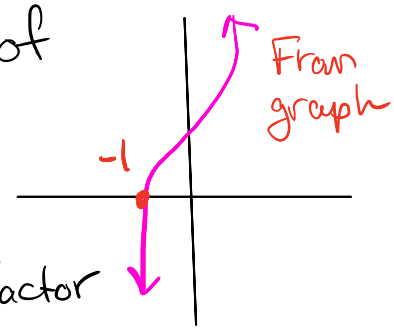
$$f(x) = x^2+1 = (x+i)(x-i) = 0$$

i is a root so $x-i$ is a factor

$-i$ is a root so $x-(-i)$ is a factor

Ex Find the roots (and factors) of

$$f(x) = x^3+1 = 0$$



$x=-1$ is a root $\Rightarrow$ $(x-(-1))$ is a factor

$(x+1)$

$$f(x) = (x+1)(???) = 0$$

$$\begin{array}{r|l} & x^2 - x + 1 \\ \hline x+1 & x^3+0x^2+0x+1 \\ & -(x^3+x^2) \\ \hline & -x^2+0x \\ & -(-x^2-x) \\ \hline \end{array}$$

$$\begin{array}{r} x+1 \\ -(x+1) \\ \hline 0 \end{array}$$

$$f(x) = x^3+1 = \underbrace{(x+1)}_{\text{prime}}\underbrace{(x^2-x+1)}_{\text{needs factoring!}} = 0$$

use quadratic formula!

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2\cdot a}$$

$$x^2-x+1=0$$

$a=1 \quad b=-1 \quad c=1$

$$x = \frac{-(-1) \pm \sqrt{(-1)^2-4(1)\cdot(1)}}{2(1)} = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1 \pm \sqrt{-3}}{2}$$

$$= \frac{1}{2} \pm \frac{i\sqrt{3}}{2}$$

$$\boxed{\begin{aligned} \sqrt{-3} &= \sqrt{-1\cdot 3} \\ &= \sqrt{-1}\cdot\sqrt{3} \\ &= i\sqrt{3} \end{aligned}}$$

$\frac{1}{2} + i\frac{\sqrt{3}}{2}$ is a root so $\left(x - \left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)\right)$ is a factor

$\frac{1}{2} - i\frac{\sqrt{3}}{2}$ is a root so $\left(x - \left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)\right)$ is a factor

$$f(x) = x^3+1 = (x+1)\left[\left(x - \left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)\right)\left(x - \left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)\right)\right] = 0$$

$$\boxed{\text{roots} \rightarrow -1,\ \frac{1}{2} + i\frac{\sqrt{3}}{2}\,^{*},\ \frac{1}{2} - i\frac{\sqrt{3}}{2}\,^{*}}$$

* * are called "complex conjugates"

Observation:

1) Every polynomial of degree n can be factored as

$$f(x) = m(x-c_1)(x-c_2)\cdots(x-c_n).$$

2) Every polynomial of degree n has at most n roots.

3) The factor $(x-c)$ for a root c could appear multiple times. E $(x-c)^{k}$

K is called the <u>multiplicity</u> of c.

4) If c is a complex root then so is its complex conjugate.