Permutations

In this section we will develop an even faster way to solve some of the problems we have already learned to solve by other means. Let's start with a couple examples.

Example 25

How many different ways can the letters of the word MATH be rearranged to form a four-letter code word?

M T H A ← one arrangement of letters

M A A A
A T H
T H
H

use this to determine how many arrangements exist

$\underline{4} \cdot \underline{3} \cdot \underline{2} \cdot \underline{1} = 24$

# of options for each space

using counting rule multiply all options

MATH	AHMT	TAHM	HATM
MAHT	AHTM	TAMH	HAMT
MHAT	ATMH	THAM	HMAT
MHTA	ATHM	THMA	HMTA
MTAH	AMHT	TMHA	HTAM
MTHA	AMTH	TMAH	HTMA

24 total outcomes
24 total arrangements for MATH

number of ways n objects can be arranged

Factorial
$n! = n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1$

n is a whole number.

$n \in \{0, 1, 2, 3, \ldots\}$

Example 26

How many ways can five different door prizes be distributed among five people?

$5! = \underline{5} \cdot \underline{4} \cdot \underline{3} \cdot \underline{2} \cdot \underline{1} = 120$ different distributions

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$7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

$= 7 \cdot 6!$

$7! = 7 \cdot 6 \cdot 5!$

$11! = 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7!$

$1! = 1$

The case of $0!$

$3! = 3 \cdot 2 \cdot 1$

$2! = 2 \cdot 1 = \frac{3!}{3}$

$1! = 1 = \frac{2!}{2}$

$0! = \frac{1!}{1} = \frac{1}{1} = 1$

$0! = 1$

→ There is one arrangement for 0 objects

Example 27

A charity benefit is attended by 25 people and three gift certificates are given away as door prizes: one gift certificate is in the amount of \$100, the second is worth \$25 and the third is worth \$10. Assuming that no person receives more than one prize, how many different ways can the three gift certificates be awarded?

no replacement

25 24 23

~~\$100~~ ~~\$25~~ ~~\$10~~

1st 2nd 3rd

25 People

3 gift certificates

different values indicate order matters

$$_{25}P_3 = 25 \cdot 24 \cdot 23 = 13{,}800 \text{ ways}$$

$$_{25}P_3 = \frac{25!}{(25-3)!} = \frac{25!}{22!} = \frac{25 \cdot 24 \cdot 23 \cdot \cancel{22!}}{\cancel{22!}} = 13{,}800 \text{ ways}$$

Example 28

Eight sprinters have made it to the Olympic finals in the 100-meter race. In how many different ways can the gold, silver and bronze medals be awarded?

no replacement

order matters

$$\underset{G}{8} \quad \underset{S}{7} \quad \underset{B}{6}$$

$$_8P_3 = 8 \cdot 7 \cdot 6 = 336 \text{ different ways}$$

$$_8P_3 = \frac{8!}{(8-3)!} = \frac{8!}{5!} = \frac{8 \cdot 7 \cdot 6 \cdot \cancel{5!}}{\cancel{5!}}$$

$$= 336$$

Permutations

$_nP_r = n \cdot (n-1) \cdot (n-2) \cdots (n-r+1)$

We say that there are $_nP_r$ **permutations** of size r that may be selected from among n choices *without replacement* when *order matters*.

It turns out that we can express this result more simply using factorials.

$$_nP_r = \frac{n!}{(n-r)!}$$

arrangements of n items into r slots

Example 29

I have nine paintings and have room to display only four of them at a time on my wall. How many different ways could I do this?

implies order matters
no replacement
different arrangements

$$_9P_4 = \frac{9!}{(9-4)!} = \frac{9!}{5!} = \frac{9\cdot 8\cdot 7\cdot 6\cdot \cancel{5!}}{\cancel{5!}} = \boxed{3024 \text{ different ways}}$$

$$_9P_4 = \underline{9}\cdot\underline{8}\cdot\underline{7}\cdot\underline{6} = 3024 \text{ different ways}$$

9 objects
4 spots

Example 30

How many ways can a four-person executive committee (president, vice-president, secretary, treasurer) be selected from a 16-member board of directors of a non-profit organization?

order matters
no replacement

16 members
4 spots

$$_{16}P_4 = \frac{16!}{(16-4)!} = \frac{16!}{12!} = \frac{16\cdot 15\cdot 14\cdot 13\cdot \cancel{12!}}{\cancel{12!}} = 43{,}680 \text{ ways}$$

$$_{16}P_4 = \underline{16}\cdot\underline{15}\cdot\underline{14}\cdot\underline{13} = 43{,}680 \text{ ways}$$

Try it Now 7

How many 5 character passwords can be made using the letters A through Z

a. if repeats are allowed

b. if no repeats are allowed

replacement is possible

a.) $\underline{26}\cdot\underline{26}\cdot\underline{26}\cdot\underline{26}\cdot\underline{26} = 26^5 = 11{,}881{,}376$

b.) $\underline{26}\cdot\underline{25}\cdot\underline{24}\cdot\underline{23}\cdot\underline{22} = 7{,}893{,}600$

$$_{26}P_5 = \frac{26!}{(26-5)!} = \frac{26!}{21!} = \frac{26\cdot 25\cdot 24\cdot 23\cdot 22\cdot \cancel{21!}}{\cancel{21!}} =$$