Recall Permutations

- arrangement of items in a specific number of places

- $_nP_r = \dfrac{n!}{(n-r)!} = \underbrace{\underline{n} \cdot \underline{(n-1)}\,\underline{(n-2)}\,\underline{(n-3)} \cdots}_{r \text{ places}}$

- no replacement

- order matters ×

But what if order didn't matter?

Example 31

A charity benefit is attended by 25 people at which three \$50 gift certificates are given away as door prizes. Assuming no person receives more than one prize, how many different ways can the gift certificates be awarded?

If order did matter,

$n = 25$

$r = 3$

$_{25}P_3 = \underline{25} \cdot \underline{24} \cdot \underline{23}$

$$_{25}P_3 = \frac{25!}{(25-3)!} = \frac{25!}{22!} = \frac{25 \cdot 24 \cdot 23 \cdot \cancel{22!}}{\cancel{22!}}$$

$= 13{,}800$ ways

e.g. Jenna, Ramone, Sean all won.

JRS RJS SRJ

JSR RSJ SJR

Possible permutations 6 permutations

$$_3P_3 = \frac{3!}{(3-3)!} = \frac{3!}{1} = 3!$$

or $$_3P_3 = 3 \cdot 2 \cdot 1 = 3!$$

If order did not matter, instead of arrangements/permutations, we would have combinations

$$_nC_r = \frac{_nP_r}{_rP_r} = \frac{_nP_r}{r!} = \frac{\frac{n!}{(n-r)!}}{r!}$$

$$= \frac{n!}{(n-r)!} \div \frac{r!}{1}$$

$$= \frac{n!}{(n-r)!} \cdot \frac{1}{r!}$$

$$_nC_r = \frac{n!}{r!(n-r)!}$$

Example 31

> A charity benefit is attended by 25 people at which three $50 gift certificates are given away as door prizes. Assuming no person receives more than one prize, how many different ways can the gift certificates be awarded?

But order doesn't matter

$${}_{25}C_3 = \frac{{}_{25}P_3}{3!}$$

${}_{25}P_3$ ← total value if order did matter

$3!$ ← arrangements in 3 spots

$$= \frac{13800}{3\cdot 2\cdot 1} = \frac{13800}{6} = 2300 \text{ possible combinations}$$

Using formula

$${}_{25}C_3 = \frac{25!}{3!\,(25-3)!}$$

$$= \frac{25!}{3!\,22!}$$

$$= \frac{25\cdot 24\cdot 23\cdot \cancel{22!}}{3\cdot 2\cdot 1\cdot \cancel{22!}}$$

$$= \frac{25\cdot \cancel{24}^{12}\cdot 23}{3\cdot \cancel{2}_{1}\cdot 1}$$

$\frac{24}{2} = \frac{12}{1}$

$$= \frac{25\cdot \cancel{12}^{4}\cdot 23}{\cancel{3}_{1}}$$

$\frac{12}{3} = \frac{4}{1}$

$$= 25\cdot 4\cdot 23 = 2300 \text{ combinations}$$

$${}_{25}C_3 = \frac{25!}{3!\,(25-3)!}$$

$$= \frac{25!}{3!\,22!}$$

$$= \frac{25\cdot 24\cdot 23\cdot \cancel{22!}}{3\cdot 2\cdot 1\cdot \cancel{22!}}$$

$$= \frac{25\cdot \cancel{24}^{4}\cdot 23}{\cancel{6}_{1}}$$

$$\boxed{\frac{24}{6} = \frac{4}{1}}$$

$$= 25\cdot 4\cdot 23$$

$$= 2300$$

Combinations

$${}_nC_r = \frac{{}_nP_r}{{}_rP_r}$$

We say that there are ${}_nC_r$ **combinations** of size r that may be selected from among n choices *without replacement* where *order doesn't matter*.

We can also write the combinations formula in terms of factorials:

$${}_nC_r = \frac{n!}{(n-r)!r!}$$

nCr

Example 32

A group of four students is to be chosen from a 35-member class to represent the class on the student council. How many ways can this be done?

no replacement

$n = 35$

$r = 4$

no specificity on student council

→ order doesn't matter

$${}_{35}C_4 = \frac{{}_{35}P_4}{4!} = \frac{35 \cdot 34 \cdot \cancel{33}^{11} \cdot \cancel{32}^{\cancel{16}\,4}}{1\,\cancel{4} \cdot \cancel{3}_1 \cdot \cancel{2}_1 \cdot 1}$$

$$= 35 \cdot 34 \cdot 11 \cdot 4$$

$= 52,360$ combinations of four students

$$_{35}C_4 = \frac{35!}{4!(35-4)!} = \frac{35!}{4!\,31!}$$

$$= \frac{35 \cdot 34 \cdot 33 \cdot 32 \cdot \cancel{31!}}{4 \cdot 3 \cdot 2 \cdot 1 \cdot \cancel{31!}}$$

$$= \frac{35 \cdot 34 \cdot \cancel{33}^{11} \cdot \cancel{32}^{4}}{\cancel{8}_{1} \cdot \cancel{3}}$$

$$\frac{32}{8} = 4$$

$$\frac{33}{3} = 11$$

$$= 52{,}360 \text{ combinations}$$

Try it Now 8

The United States Senate Appropriations Committee consists of 29 members; the Defense Subcommittee of the Appropriations Committee consists of 19 members. Disregarding party affiliation or any special seats on the Subcommittee, how many different 19-member subcommittees may be chosen from among the 29 Senators on the Appropriations Committee?

$_nC_r$

$n = 29$

$r = 19$

$$_{29}C_{19} = \frac{29!}{19!(29-19)!} = \frac{29!}{19! \cdot 10!}$$

$$= \frac{29 \cdot 28 \cdot 27 \cdot 26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 \cdot 21 \cdot 20 \cdot \cancel{19!}}{\cancel{19!} \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$$

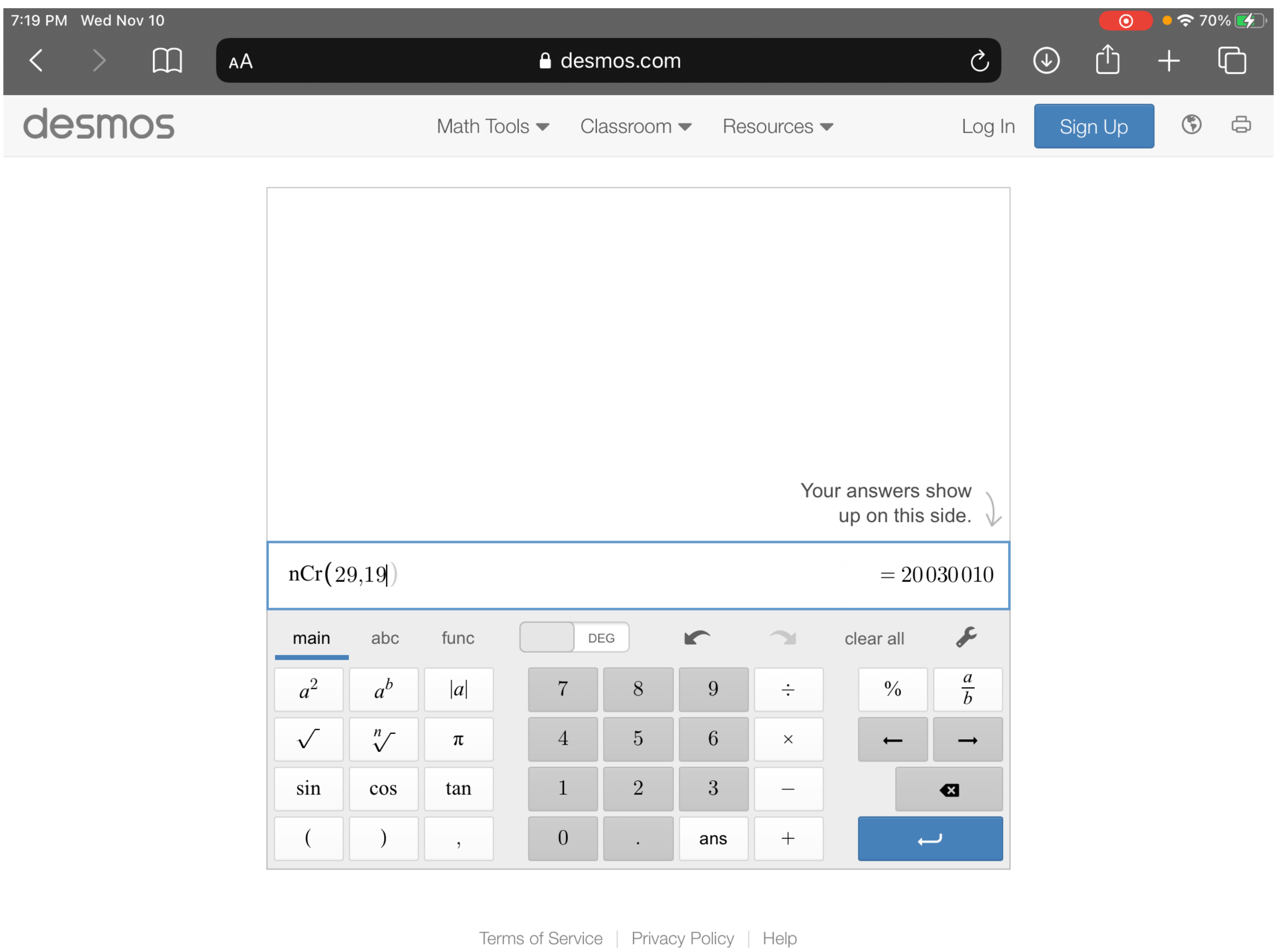
$$= \frac{29\cdot 28\cdot \cancel{27}\cdot 26\cdot 25\cdot \cancel{24}\cdot 23\cdot 22\cdot 21\cdot \cancel{20}}{\cancel{10}\cdot \cancel{9}\cdot 8\cdot 7\cdot \cancel{6}\cdot 5\cdot \cancel{4}\cdot \cancel{3}\cdot \cancel{2}\cdot 1}$$

$$= \frac{29\cdot \overset{4}{\cancel{28}}\cdot 26\cdot \overset{5}{\cancel{25}}\cdot 23\cdot 22\cdot 21}{8\cdot \underset{1}{\cancel{7}}\cdot \underset{1}{\cancel{5}}}$$

$$= \frac{29\cdot \cancel{4}\cdot 26\cdot 5\cdot 23\cdot \overset{11}{\cancel{22}}\cdot 21}{\cancel{4}\cdot \underset{1}{\cancel{2}}}$$

$$= 29\cdot 26\cdot 5\cdot 23\cdot 11\cdot 21$$

$= 20{,}030{,}010$ combinations



Example 33

The United States Senate Appropriations Committee consists of 29 members, 15 Republicans and 14 Democrats. The Defense Subcommittee consists of 19 members, 10 Republicans and 9 Democrats. How many different ways can the members of the Defense Subcommittee be chosen from among the 29 Senators on the Appropriations Committee?

↳ Choosing from 2 different sets

Republican

$n = 15$

$r = 10$

$$_{15}C_{10} = \frac{15!}{10!\,(15-10)!}$$

$$= \frac{15!}{10!\,5!}$$

$$= \frac{\cancel{15}\cdot\cancel{14}^{7}\cdot 13\cdot\cancel{12}^{3}\cdot 11\cdot\cancel{10!}}{\cancel{5}\cdot\cancel{4}_{1}\cdot\cancel{3}\cdot\cancel{2}_{1}\cdot 1\cdot\cancel{10!}}$$

$$= 7\cdot 13\cdot 3\cdot 11$$

$= 3{,}003$ combinations of Republicans

Democratic

$n = 14$

$r = 9$

$$_{14}C_{9} = \frac{14!}{9!\,(14-9)!}$$

$$= \frac{14!}{9!\cdot 5!}$$

$$= \frac{14\cdot 13\cdot\cancel{12}\cdot 11\cdot\cancel{10}\cdot\cancel{9!}}{\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot 1\cdot\cancel{5!}}$$

$$= 14\cdot 13\cdot 11$$

$= 2{,}002$ combinations of Democrats

By Counting Principle $= \underset{R}{(3003)}\cdot\underset{D}{(2002)} = 6{,}012{,}006$ combinations

multiply the two.

$$= {}_{15}C_{10}\cdot{}_{14}C_{9} =$$