Quantum teleportation

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For the quantum teleportation, the sceuatio is that Alice wishes to communicate the state of a qubit to Bob. The goal of teleportation is to transfer the unknown state information of the source qubit without measuring or observing, to the destination qubit, thereby avoiding the disturbance of the source qubit.

Note that the process is not faster than light.

Suppose Alice has only a classical channel linking her to Bob. Teleportation is a protocol which allows Alice to communicate state of the qubit exactly to Bob, sending only two bits of classical information to him. Like superdense coding Alice and Bob initially share the Bell state

$$|\beta_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle. \qquad (1)$$

Let's consider the teleportation protocol step by step.

1. At start assume Alice has a single-qubit state

$$|\Psi\rangle = \alpha|0\rangle + \beta|1\rangle \qquad (2)$$

and α and β in the state are unknown. Therefore, the necessary information to specify the state at Alice location are not available

2. The shared entangled Bell state (1) has Alise (the first qubit) and Bob (second qubit)

Therefore, the first half of the Bell state has Alice and the second half of the Bell state has Bob. Thus, Alice has two qubits the state $|\varphi\rangle$ and the first half of Bell state and Bob has the second half of Bell state.

3. To teleport the qubit at. Alice lalation to Bob, create a tensor product of qubit $|\varphi\rangle$ and $|\beta_{00}\rangle$

$$|\varphi_1\rangle = |\varphi\rangle \otimes |\beta_{00}\rangle =$$

$$= (\alpha|0\rangle + \beta|1\rangle) \otimes \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) =$$

$$= \frac{\alpha|0\rangle \otimes |00\rangle}{\sqrt{2}} + \frac{\alpha|0\rangle \otimes |11\rangle}{\sqrt{2}} + \frac{\beta|1\rangle \otimes |00\rangle}{\sqrt{2}} + \beta|0\rangle \otimes |11\rangle$$

$$= \frac{1}{\sqrt{2}}(\alpha|000\rangle + \alpha|011\rangle + \beta|100\rangle + \beta|111\rangle)$$

therefore, there are three qubits at the start:

i) qubit 1 is in an unknow state that Alice possesses that should be teleported to Bob

ii) Qubit 2 is the first half of the Bell state that Alice possesses also.

iii) Qubit 3 is the second half of Bell state that Bob possesses.

4. Two qubits that Alice possesses she send via controlled-NOT (X) gate. as we know X gate inverts the state of the second qubit if the first is in state 1,

and doing nothing if the first state is 0
Therefore, we have

$$|\varphi_2\rangle = X|\varphi_1\rangle = X\frac{1}{\sqrt{2}}\left(\alpha|000\rangle + \alpha|011\rangle + \beta|100\rangle + \beta|111\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(\alpha|000\rangle + \alpha|011\rangle + \beta|110\rangle + \beta|101\rangle\right) \quad (3)$$

5. The First qubit that should be teleportated is send through Hadamard gate. The first qubit in Eq. (3) is being in state $|0\rangle, |0\rangle, |1\rangle, |1\rangle$. Therefore, the Hadamard gate transforming $|0\rangle$ and $|1\rangle$ state into.

$$H|0\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|\right)|0\rangle = $$
$$= \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \quad (4)$$

$$H|1\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|\right)|1\rangle = $$
$$= \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right) \quad (5)$$

$$|\varphi_3\rangle = H|\varphi_2\rangle = H\frac{1}{\sqrt{2}}\left(\alpha|000\rangle + \alpha|011\rangle + \beta|110\rangle + \beta|101\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(\frac{\alpha}{\sqrt{2}}(|0\rangle + |1\rangle)|00\rangle + \frac{\alpha}{\sqrt{2}}(|0\rangle + |1\rangle)|11\rangle + \frac{\beta}{\sqrt{2}}(|0\rangle - |1\rangle)|10\rangle + \frac{\beta}{\sqrt{2}}(|0\rangle - |1\rangle)|01\rangle\right) =$$

(Notice that by regrouping the qubits we always keeping them in the same order)

$$= \frac{\alpha}{2}\left(|000\rangle + |100\rangle + |011\rangle + |111\rangle\right) +$$

$$\frac{\beta}{2}\left(|010\rangle - |110\rangle + |001\rangle - |101\rangle\right) =$$

$$= \frac{1}{2}|00\rangle(\alpha|0\rangle + \beta|1\rangle) + \frac{1}{2}|01\rangle\left(\alpha|1\rangle + \beta|0\rangle\right) +$$

$$+ \frac{1}{2}|10\rangle\left(\alpha|0\rangle - \beta|1\rangle\right) + \frac{1}{2}|11\rangle(\alpha|1\rangle - \beta|0\rangle) \quad (6)$$

at this stage of transformation the first and the second qubit posesses Alice while the third qubit has Bob. Using the identity operator and Pauli operators X and Z the state $|\varphi_3\rangle$ given by Eq. (6) can be rewritten as

(Please be advised that $|\Psi\rangle = \alpha|0\rangle + \beta|1\rangle$

$$I|\Psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

$$X|\Psi\rangle = (|0\rangle\langle 1| + |1\rangle\langle 0|)(\alpha|0\rangle + \beta|1\rangle) =$$

$$= \alpha|1\rangle + \beta|0\rangle$$

$$Z|\Psi\rangle = (|0\rangle\langle 0| - |1\rangle\langle 1|)(\alpha|0\rangle + \beta|1\rangle) =$$

$$= \alpha|0\rangle - \beta|1\rangle.$$

$$XZ|\Psi\rangle = (|0\rangle\langle 1| + |1\rangle\langle 0|)(|0\rangle\langle 0| - |1\rangle\langle 1|)(\alpha|0\rangle + \beta|1\rangle)$$

$$= (|0\rangle\langle 1| + |1\rangle\langle 0|)(\alpha|0\rangle - \beta|1\rangle) =$$

$$= \alpha|1\rangle - \beta|0\rangle$$

Using above mentioned operator Eq. (6) can be rewritten in more convinient form;

$$|\varphi_2\rangle = \frac{1}{2}|00\rangle I|\Psi\rangle + \frac{1}{2}|01\rangle X|\Psi\rangle + \frac{1}{2}|10\rangle Z|\Psi\rangle + \frac{1}{2}|11\rangle XZ|\Psi\rangle. \quad (7)$$

Important to note that qubit 1 and 2 are different in each term of Eq. (7) and if qubits in location where is Alice is measured the outcomes can be encoded using one of the following pairs of classical bits

00, 01, 10, 11.

This means such measurement has impact on qubit 3 that possesses Bob. In particular in Table I we have listed four possible states of the third qubit ~~that~~ after measurement

measured bits	States that Bob has
00	$\alpha|0\rangle + \beta|1\rangle$
01	$\alpha|1\rangle + \beta|0\rangle$
10	$\alpha|0\rangle - \beta|1\rangle$
11	$\alpha|1\rangle - \beta|0\rangle$

The possible outcomes are given in Table 1 that Bob possesses uppon measuring two qubits by Alice. Bob has to apply a unitary transformation on the state of the third qubit so that the state of qubit 1 and qubit 2 are restored to their original value

I <u>State $|00\rangle$</u>, qubit 3 in state $\alpha|0\rangle+\beta|1\rangle$
unitary operator I is applied to qubit 3.

$$I(\alpha|0\rangle+\beta|1\rangle = \alpha|0\rangle+\beta|1\rangle = |\Psi\rangle$$

II state $|01\rangle$ qubit 3 in state $\alpha|1\rangle+\beta|0\rangle$
unitary operator X is applied to third qubit $\alpha|1\rangle+\beta|0\rangle$

$$X(\alpha|1\rangle+\beta|0\rangle) = (|0\rangle\langle 1| + |1\rangle\langle 0|)(\alpha|1\rangle+\beta|0\rangle) =$$
$$= \alpha|0\rangle+\beta|1\rangle = |\Psi\rangle$$

III state $|10\rangle$ qubit 3 in state $\alpha|0\rangle-\beta|1\rangle$
Unitary operator Z is applied to the third qubit $\alpha|0\rangle-\beta|1\rangle$

$$Z(\alpha|0\rangle-\beta|1\rangle) = (|0\rangle\langle 0| - |1\rangle\langle 1|)(\alpha|0\rangle-\beta|1\rangle) =$$
$$= \alpha|0\rangle+\beta|1\rangle = |\Psi\rangle$$

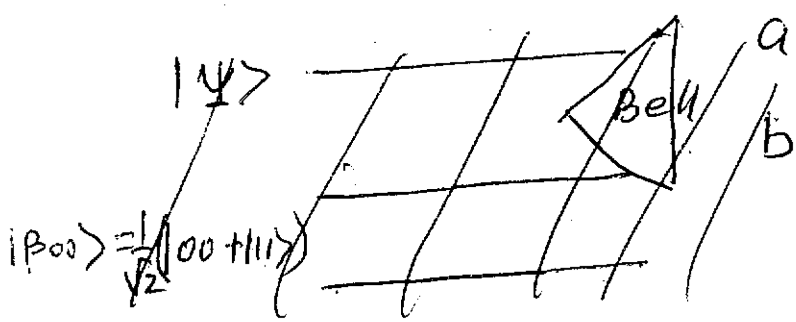
IV state $|11\rangle$ qubit 3 is in state $\alpha|1\rangle-\beta|0\rangle$
Unitary operator ZX is applied to the third qubit $\alpha|1\rangle-\beta|0\rangle$

$$ZX(\alpha|1\rangle-\beta|0\rangle) = Z(\alpha|0\rangle-\beta|1\rangle) =$$
$$= \alpha|0\rangle+\beta|1\rangle = |\Psi\rangle$$

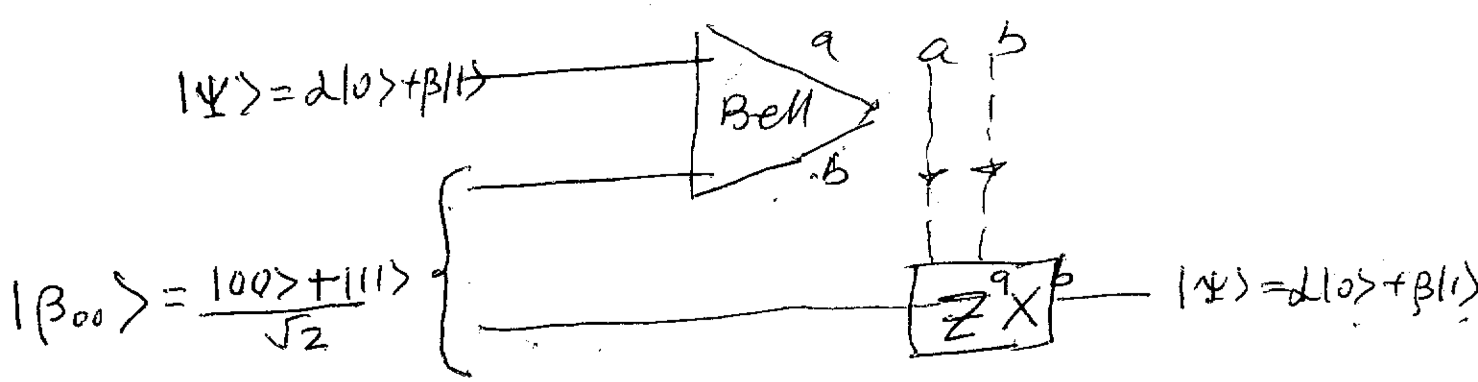
Therefore it is show that in each of the above cases there is a unitary transformation that restores the state of the qubit in Bob's Location to the original state $|\Psi\rangle = \alpha|0\rangle + \beta|1\rangle$.

Table II has the summary of these unitary transformations

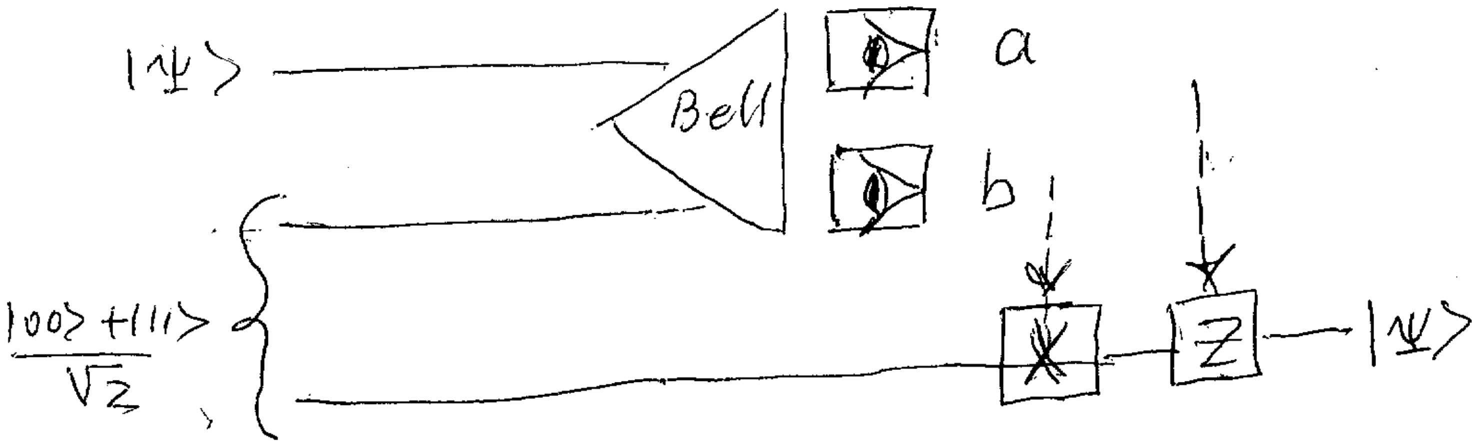
States that Alice has	Unitary transformations that Bob perfoms
00	I
01	X
10	Z
11	ZX



Bob performs a unitary transformation $Z^b X^a$ depending on the classical bits a, b that Alice sent him.



The circuit for quantum teleportation is shown below



Alice performed a joint measurement of $|\Psi\rangle$ by her and shared ~~EPR $|\beta_{00}\rangle$ by her~~ half of EPR pair with Bob. She sent the result of the measurement which are classical a and b to Bob over classical channel. The value a and b, Bob used to control the operation on his qubit. After Bob performed unitary ~~transphor~~ transformation, his qubit is left in the state $|\Psi\rangle$

Home work:
Show that

$$|\Psi\rangle|\beta_{00}\rangle = \frac{1}{2}|\beta_{00}\rangle|\Psi\rangle + \frac{1}{2}|\beta_{01}\rangle X|\Psi\rangle + \frac{1}{2}|\beta_{10}\rangle Z|\Psi\rangle + \frac{1}{2}|\beta_{11}\rangle XZ|\Psi\rangle$$