Chapter 6. Superdense Coding

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1 Basis Change Circuit

The quantum state can be measured with respect of different basis. Suppose we want to make measurement of 2-qubit state with respect to orthogonal basis

$$\begin{aligned} |\beta_{00}\rangle &= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \qquad |\beta_{01}\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \\ |\beta_{11}\rangle &= \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \qquad |\beta_{10}\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle). \end{aligned} \tag{1}$$

This basis is known as the Bell basis and the four states $|\beta_{00}\rangle, |\beta_{01}\rangle, |\beta_{10}\rangle, |\beta_{11}\rangle$ are known as the Bell states. These states also known as Einstein-Podolsky-Rosen (EPR) states. Let have as an input the state $|\varphi_1\rangle = |0\rangle|0\rangle = |00\rangle$ and consider that this state passes through the circuit shown in Fig. 1. After the Hadamard gate the state is

$$|\varphi_2\rangle = H|00\rangle = \frac{1}{\sqrt{2}}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|)|0\rangle|0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle). \tag{2}$$

Above we maintained the order of the qubits. Now let's apply the controlled-NOT gate X to the state $|\varphi_2\rangle$:

$$\begin{aligned} |\varphi_3\rangle &= H|00\rangle = X|\varphi_2\rangle = (|0\rangle\langle 1| + |1\rangle\langle 0|)\frac{1}{\sqrt{2}}(|00\rangle + |10\rangle) \\ &= (|0\rangle\langle 1| + |1\rangle\langle 0|)\frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|0\rangle) = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle). \end{aligned} \tag{3}$$

Therefore we have $|\varphi_3\rangle = |\beta_{00}\rangle$ and the circuit in Fig. 1 performs the basis change from the computational one to the Bell basis. Similarly the circuit performs the remaining three computational states $|0\rangle|1\rangle = |01\rangle, |1\rangle|0\rangle = |10\rangle$, $|1\rangle|1\rangle = |11\rangle$ transforming them to the $|\beta_{01}\rangle, |\beta_{10}\rangle, |\beta_{11}\rangle$, respectively.

Exercise

Implement circuit in Fig. 1 to transform the computational states $|01\rangle, |10\rangle$, $|11\rangle$ to the Bell's states $|\beta_{01}\rangle, |\beta_{10}\rangle, |\beta_{11}\rangle$.

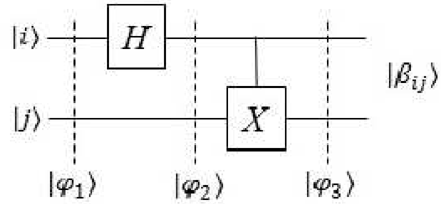

Figure 1: A circuit for basis change from the computational basis to the Bell basis

2 Superdense Coding

The superdense coding protocol idea is transmit two classical bits of information by sending a single qubit through the quantum channel. Let us consider a situation where two people Alice and Bob are located in different parts of the world. Alice has two bits of information a and b and could send two qubits with the message incoded in them. However, she would like to communicate these two bits to Bob by sending him just a single qubit. This option requires that Alice and Bob initially share a pair of entangled qubits. Assume that initially Alice and Bob share one of the Bell state. Let say they share the state $|\beta_{00}\rangle$:

$$|\beta_{00}\rangle = \frac{1}{\sqrt{2}} \left(|00\rangle + |11\rangle \right).$$

Alice took the first qubit in each term and Bob took the second qubit (maintain the order of the qubits). Hence, Alice and Bob each posses half of the Bell state, that is they share one unit of the entangled pair. Alice performs one of the four 1-qubit unitary gates, depending on the 2 classical bits that she wants to communicate to Bob. Alice wants to send the one of the possible four combinations 00, 01, 10, 11. She will apply the operation $U_{ab} = Z^a X^b$ (ab is 00, 01, 10, 11, respectively, depending on the bits she would like to send) on the qubit she possesses and applies the identity operator I on the qubit that Bob is possessed. Below are presented all cases depending which classical bits Alice wants to send.

Case 00. Classical bit to be sent 00. Nothing needs to be done, so Alice applies unitary operator on her part of the Bell state

$$\begin{aligned} 00 \quad &: \quad I \otimes I \left|\beta_{00}\right\rangle = \left(\left|0\right\rangle\left\langle 0\right| + \left|1\right\rangle\left\langle 1\right|\right) \otimes \left(\left|0\right\rangle\left\langle 0\right| + \left|1\right\rangle\left\langle 1\right|\right) \frac{1}{\sqrt{2}}\left(\left|00\right\rangle + \left|11\right\rangle\right) \\ &= \left(\left|0\right\rangle\left\langle 0\right| + \left|1\right\rangle\left\langle 1\right|\right) \frac{1}{\sqrt{2}}\left(\left|00\right\rangle + \left|11\right\rangle\right) = \frac{1}{\sqrt{2}}\left(\left|00\right\rangle + \left|11\right\rangle\right) = \left|\beta_{00}\right\rangle. \end{aligned} \tag{4}$$

After the application of unitary operation Alice sends her half of the entangled qubits q_0 to Bob. Bob combines it with his q_1 applying the controlled-NOT operator to the pair $(q_0 q_1)$ assuming that Alice's qubit q_0 is a control bit. Next a Hadamard transform on the first qubit of the pair is applied, which leads to the unentanglement of the Bell state and results in a unique state that corresponds to the 2-bit information send by Alice. The process can be explained as this is shown below.

The controlled-NOT (X) operator has no effect on the $\left|0\right\rangle\left|0\right\rangle$ part of the Bell state, but since $q_0 = 1$, it changes the $\left|1\right\rangle\left|1\right\rangle$ to $\left|1\right\rangle\left|0\right\rangle$:

$$\begin{aligned} X\left|\beta_{00}\right\rangle &= X\frac{1}{\sqrt{2}}\left(\left|00\right\rangle + \left|11\right\rangle\right) = \frac{1}{\sqrt{2}}\left(\left|00\right\rangle + \left|10\right\rangle\right) \\ &= \frac{1}{\sqrt{2}}\left(\left|0\right\rangle\left|0\right\rangle + \left|1\right\rangle\left|0\right\rangle\right). \end{aligned} \tag{5}$$

Bob then applies the Hadamard gate on the first q_0 qubit of the entangled pair as

$$\begin{aligned} H\frac{1}{\sqrt{2}}\left(\left|0\right\rangle\left|0\right\rangle + \left|1\right\rangle\left|0\right\rangle\right) &= \frac{1}{\sqrt{2}}\left(\left|0\right\rangle\left\langle 0\right| + \left|0\right\rangle\left\langle 1\right| + \left|1\right\rangle\left\langle 0\right| - \left|1\right\rangle\left\langle 1\right|\right)\frac{1}{\sqrt{2}}\left(\left|0\right\rangle\left|0\right\rangle + \left|1\right\rangle\left|0\right\rangle\right) \\ &= \frac{1}{2}\left(\left|0\right\rangle\left|0\right\rangle + \left|1\right\rangle\left|0\right\rangle + \left|0\right\rangle\left|0\right\rangle - \left|1\right\rangle\left|0\right\rangle\right) = \left|0\right\rangle\left|0\right\rangle = \left|00\right\rangle. \end{aligned} \tag{6}$$

Finally Bob measures both qubits and gets Alice's message 00.

Case 01: Alice applies $X \otimes I$ on the Bell state $\left|\beta_{00}\right\rangle$ to send the 01 bits

$$\begin{aligned} 01 \quad &: \quad X \otimes I \left|\beta_{00}\right\rangle = \left(\left|0\right\rangle\left\langle 1\right| + \left|1\right\rangle\left\langle 0\right|\right) \otimes \left(\left|0\right\rangle\left\langle 0\right| + \left|1\right\rangle\left\langle 1\right|\right) \frac{1}{\sqrt{2}}\left(\left|00\right\rangle + \left|11\right\rangle\right) \\ &= \left(\left|0\right\rangle\left\langle 1\right| + \left|1\right\rangle\left\langle 0\right|\right) \frac{1}{\sqrt{2}}\left(\left|00\right\rangle + \left|11\right\rangle\right) = \frac{1}{\sqrt{2}}\left(\left|01\right\rangle + \left|10\right\rangle\right) = \left|\beta_{01}\right\rangle \end{aligned} \tag{7}$$

The X operation applied by Bob changes the $\left|1\right\rangle\left|0\right\rangle$ parts of the Bell state $\left|\beta_{01}\right\rangle$:

$$\begin{aligned} X\left|\beta_{01}\right\rangle &= X\frac{1}{\sqrt{2}}\left(\left|01\right\rangle + \left|10\right\rangle\right) = \frac{1}{\sqrt{2}}\left(\left|01\right\rangle + \left|11\right\rangle\right) \\ &= \frac{1}{\sqrt{2}}\left(\left|0\right\rangle\left|1\right\rangle + \left|1\right\rangle\left|1\right\rangle\right). \end{aligned} \tag{8}$$

Bob applies now the Hadamard operator H on the first qubit of the entangled pair and this change Bob's state into

$$\begin{aligned} H\frac{1}{\sqrt{2}}\left(|1\rangle|1\rangle+|0\rangle|1\rangle\right) &= \frac{1}{\sqrt{2}}\left(|0\rangle\langle 0|+|0\rangle\langle 1|+|1\rangle\langle 0|-|1\rangle\langle 1|\right)\frac{1}{\sqrt{2}}\left(|1\rangle|1\rangle+|0\rangle|1\rangle\right) \\ &= \frac{1}{2}\left(|0\rangle|1\rangle-|1\rangle|1\rangle+|0\rangle|1\rangle+|1\rangle|1\rangle\right)=|0\rangle|1\rangle=|01\rangle . \qquad (9) \end{aligned}$$

Therefore, Bob after performing measurements gets the massage 01.

Case 10: When Alice is sending 10 bits she applies $Z\otimes I$ operator on the Bell state $|\beta_{10}\rangle$

$$\begin{aligned} 10 \quad &: \quad Z\otimes I\,|\beta_{10}\rangle\left(|0\rangle\langle 0|-|1\rangle\langle 1|\right)\otimes\left(|0\rangle\langle 0|+|1\rangle\langle 1|\right)\frac{1}{\sqrt{2}}\left(|00\rangle+|11\rangle\right) \\ &= \left(|0\rangle\langle 0|-|1\rangle\langle 1|\right)\frac{1}{\sqrt{2}}\left(|00\rangle+|11\rangle\right)=\frac{1}{\sqrt{2}}\left(|00\rangle-|11\rangle\right)=|\beta_{10}\rangle . \qquad (10) \end{aligned}$$

This results in $|\beta_{10}\rangle$ state transmitted to Bob. Bob applies X operation changes the $|1\rangle|1\rangle$ parts of the Bell state $|\beta_{10}\rangle$

$$\begin{aligned} X|\beta_{10}\rangle &= X\frac{1}{\sqrt{2}}\left(|00\rangle-|11\rangle\right)=\frac{1}{\sqrt{2}}\left(|00\rangle-|10\rangle\right) \\ &= \frac{1}{\sqrt{2}}\left(|0\rangle|0\rangle-|1\rangle|0\rangle\right). \qquad (11) \end{aligned}$$

Bob applies now Hadamard operator H on the first qubit of the entangled pair and this change the Bob state into

$$\begin{aligned} H\frac{1}{\sqrt{2}}\left(|0\rangle|0\rangle-|1\rangle|0\rangle\right) &= \frac{1}{\sqrt{2}}\left(|0\rangle\langle 0|+|0\rangle\langle 1|+|1\rangle\langle 0|-|1\rangle\langle 1|\right)\frac{1}{\sqrt{2}}\left(|0\rangle|0\rangle-|1\rangle|0\rangle\right) \\ &= \frac{1}{2}\left(|0\rangle|0\rangle+|1\rangle|0\rangle-|0\rangle|0\rangle+|1\rangle|0\rangle\right)=|1\rangle|0\rangle=|10\rangle . \qquad (12) \end{aligned}$$

After measurements Bob gets the message, which has been sent by Alice: 10

Case 11: Alice is sending 11 classical bits of information by applying $XZ\otimes I$ on the Bell state$|\beta_{00}\rangle$:

$$\begin{aligned} 11 \quad &: \quad XZ\otimes I\,|\beta_{00}\rangle=\left(|0\rangle\langle 1|+|1\rangle\langle 0|\right)\left(|0\rangle\langle 0|-|1\rangle\langle 1|\right)\otimes\left(|0\rangle\langle 0|+|1\rangle\langle 1|\right)\frac{1}{\sqrt{2}}\left(|00\rangle+|11\rangle\right) \\ &= \left(|0\rangle\langle 1|+|1\rangle\langle 0|\right)\left(|0\rangle\langle 0|-|1\rangle\langle 1|\right)\frac{1}{\sqrt{2}}\left(|00\rangle+|11\rangle\right)= \\ &= \left(|1\rangle\langle 0|-|0\rangle\langle 1|\right)\frac{1}{\sqrt{2}}\left(|00\rangle-|11\rangle\right)=\frac{1}{\sqrt{2}}\left(|01\rangle-|10\rangle\right)=|\beta_{11}\rangle \qquad (13) \end{aligned}$$

After Bob acts by X operation that changes the $|1\rangle|1\rangle$ parts of the Bell state $|\beta_{11}\rangle$ as

$$X|\beta_{11}\rangle = X\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) = \frac{1}{\sqrt{2}}(|01\rangle - |11\rangle) = \frac{1}{\sqrt{2}}(|0\rangle|1\rangle - |1\rangle|1\rangle). \quad (14)$$

Bob applies now the Hadamard operator H on the first qubit of the entangled pair and this change the Bob state into

$$H\frac{1}{\sqrt{2}}(|0\rangle|1\rangle - |1\rangle|1\rangle) = \frac{1}{\sqrt{2}}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|)\frac{1}{\sqrt{2}}(|0\rangle|1\rangle - |1\rangle|1\rangle) = \frac{1}{2}(|0\rangle|1\rangle + |1\rangle|1\rangle - |0\rangle|1\rangle + |1\rangle|1\rangle) = |1\rangle|1\rangle = |11\rangle. \quad (15)$$

After measurement Bob gets the message, which has been sent by Alice: 11.

This process can be summarized as shown in the Table below.

Table 1: Transmission and reception of one of four possible 2-bits strings via a single qubit

Alice operation to send the classical bits			
Bits to be sent	Initial State	Unitary operation	Final state
00	$\frac{\|00\rangle+\|11\rangle}{\sqrt{2}}$	$I \otimes I$	$\frac{\|00\rangle+\|10\rangle}{\sqrt{2}}$
01	$\frac{\|00\rangle+\|11\rangle}{\sqrt{2}}$	$X \otimes I$	$\frac{\|01\rangle+\|10\rangle}{\sqrt{2}}$
10	$\frac{\|00\rangle+\|11\rangle}{\sqrt{2}}$	$Z \otimes I$	$\frac{\|00\rangle-\|11\rangle}{\sqrt{2}}$
11	$\frac{\|00\rangle+\|11\rangle}{\sqrt{2}}$	$XZ \otimes I$	$\frac{\|01\rangle-\|10\rangle}{\sqrt{2}}$
Bob's operation to recieve qubits and measure information			
Received state	Application of X	Application of H	Bits to be received
$\frac{\|00\rangle+\|10\rangle}{\sqrt{2}}$	$\frac{\|0\rangle\|1\rangle+\|1\rangle\|1\rangle}{\sqrt{2}}$	$\|0\rangle\|0\rangle = \|00\rangle$	00
$\frac{\|01\rangle+\|10\rangle}{\sqrt{2}}$	$\frac{\|0\rangle\|0\rangle+\|1\rangle\|0\rangle}{\sqrt{2}}$	$\|0\rangle\|1\rangle = \|01\rangle$	01
$\frac{\|00\rangle-\|11\rangle}{\sqrt{2}}$	$\frac{\|0\rangle\|0\rangle-\|1\rangle\|0\rangle}{\sqrt{2}}$	$\|1\rangle\|0\rangle = \|10\rangle$	10
$\frac{\|01\rangle-\|10\rangle}{\sqrt{2}}$	$\frac{\|0\rangle\|1\rangle-\|1\rangle\|1\rangle}{\sqrt{2}}$	$\|1\rangle\|1\rangle = \|11\rangle$	11