Chapter 8. Quantum Algorithms

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0.1 Exercise

Describe the effect of the CNOT gate with respect the following basis:

a) $\left\{|0\rangle \frac{1}{\sqrt{2}}(|0\rangle+|1\rangle);\ |0\rangle \frac{1}{\sqrt{2}}(|0\rangle-|1\rangle);\ |1\rangle \frac{1}{\sqrt{2}}(|0\rangle+|1\rangle);\ |1\rangle \frac{1}{\sqrt{2}}(|0\rangle-|1\rangle).\right\}$

b) $\left\{\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle);\ \frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)\frac{1}{\sqrt{2}}(|0\rangle-|1\rangle);\ \frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle);\ \frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)\frac{1}{\sqrt{2}}(|0\rangle-|1\rangle).\right\}$

1 Phase Kick-Back Algorithm

First let show that $\frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)$ is the eigenstate of the operators X and I :

$$X\frac{|0\rangle-|1\rangle}{\sqrt{2}} = (|0\rangle\langle 1|+|1\rangle\langle 0|)\frac{|0\rangle-|1\rangle}{\sqrt{2}} = (-1)\frac{|0\rangle-|1\rangle}{\sqrt{2}}; \tag{1}$$

$$I\frac{|0\rangle-|1\rangle}{\sqrt{2}} = (|0\rangle\langle 0|+|1\rangle\langle 1|)\frac{|0\rangle-|1\rangle}{\sqrt{2}} = (+1)\frac{|0\rangle-|1\rangle}{\sqrt{2}}. \tag{2}$$

The eigenvalues are -1 and +1, correspondingly. The CNOT gate appears to do nothing to the control qubit. However it can in fact affect the control qubit just as much as it does the target qubit. let us consider the Hamadard basis and show that CNOT can effectively switch of control and target qubits

CNOT$\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right) =$

$$\begin{aligned} \text{CNOT}\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right) &= \left(\frac{|0\rangle|0\rangle-|0\rangle|1\rangle+|1\rangle|1\rangle-|1\rangle|0\rangle}{\sqrt{2}\sqrt{2}}\right) \\ &= \left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right). \end{aligned} \tag{3}$$

CNOT gate applies the NOT gate X to the target qubit if the first qubit is in state $|1\rangle$ (use Eq. (1))

$$
\begin{aligned}
\text{CNOT}\,|1\rangle \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) &= |1\rangle\, X \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) = \\
&= |1\rangle \left(|0\rangle\langle 1| + |1\rangle\langle 0|\right) \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) \\
&= (-1)\,|1\rangle \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right). \qquad (4)
\end{aligned}
$$

If the first qubit is in state $|0\rangle$ CNOT gate applies the identity gate I to the target qubit. Thus, we get (see Eq. (2))

$$
\text{CNOT}\,|0\rangle \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) = |0\rangle\, I \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) = (+1)\,|0\rangle \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right). \qquad (5)
$$

The comparison of Eqs. (4) and (5) can bring one to the conclusion: since the target qubit is in an eigenstate we can effectively treat the eigenvalue as "kicked back to the control register and Eqs. (4) and (5) can be summarized as follows:

$$
\text{CNOT}\,|b\rangle \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) = (-1)^b\,|b\rangle\, I \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right), \qquad (6)
$$

where $b \in \{0, 1\}$. Therefore, CNOT gate define the following transformation of a superposed qubit $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$

$$
\text{CNOT}\,|\psi\rangle \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) = (\alpha\,|0\rangle + \beta\,|1\rangle) \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) = (\alpha\,|0\rangle - \beta\,|1\rangle) \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right). \qquad (7)
$$

The careful analysis of the last equation shows that in this case CNOT acts the same way if one applies Z gate to the control qubit.

Now we focus on more general 2-qubit gate U_f implementing an arbitrary function $f(x)$. A function f maps $\{0, 1\}$ to $\{0, 1\}$. If it is know that $f(x)$ is either *a constant* (0 or 1 for all value of x) or *balanced* (0 for exactly half for all possible x and 1 for the other half), then the problem is to decide what type function it is. All possible mapping are shown in Table below

The given function $f(x)$ should be transformed into $|x\rangle\,|y \oplus f(x)\rangle$. Let us fix the target qubit in the state $\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$ and analyses the action of U_f on arbitrary basis state $|x\rangle$ in the control qubit:

Table 1: All possible mapping of a function $\{0,1\}$ to $\{0,1\}$

$f(x)$			$f(x)$		
0	$\to$	0	1	$\to$	0
1	$\to$	0	0	$\to$	1
0	$\to$	1	1	$\to$	1
1	$\to$	1	0	$\to$	0

$$\begin{aligned} U_f\,|x\rangle \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) &= \left(\frac{U_f\,|x\rangle\,|0\rangle - U_f\,|x\rangle\,|1\rangle}{\sqrt{2}}\right) \\ &= \left(\frac{|x\rangle\,|0 \oplus f(x)\rangle - |x\rangle\,|1 \oplus f(x)\rangle}{\sqrt{2}}\right) \\ &= |x\rangle \left(\frac{|0 \oplus f(x)\rangle - |1 \oplus f(x)\rangle}{\sqrt{2}}\right) \end{aligned} \tag{8}$$

Because the action "$\oplus f(x)$" has no effect on a single bit if $f(x) = 0$ and "$\oplus f(x)$" flips the state of the bit if $f(x) = 1$, let consider the expression $|0 \oplus f(x)\rangle - |1 \oplus f(x)\rangle$ in two cases $f(x) = 0$ and $f(x) = 1$:

$$f(x) = 0 : \quad \frac{|0 \oplus f(x)\rangle - |1 \oplus f(x)\rangle}{\sqrt{2}} = \frac{|0\rangle - |1\rangle}{\sqrt{2}}, \tag{9}$$

$$f(x) = 1 : \quad \frac{|0 \oplus f(x)\rangle - |1 \oplus f(x)\rangle}{\sqrt{2}} = \frac{|1\rangle - |0\rangle}{\sqrt{2}} = -\frac{|0\rangle - |1\rangle}{\sqrt{2}}. \tag{10}$$

The last two expression are differ by a factor of (-1) which depends on the value of function $f(x)$ and one can write these expressions as follow:

$$\frac{|0 \oplus f(x)\rangle - |1 \oplus f(x)\rangle}{\sqrt{2}} = (-1)^{f(x)} \frac{|0\rangle - |1\rangle}{\sqrt{2}}. \tag{11}$$

One can associate the factor $(-1)^{f(x)}$ with the first qubit and therefore we have

$$U_f\,|x\rangle \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) = (-1)^{f(x)}\,|x\rangle \frac{|0\rangle - |1\rangle}{\sqrt{2}}. \tag{12}$$

For the a superposed qubit $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$, we have

$$\begin{aligned} U_f\,|\psi\rangle \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) &= U_f\,(\alpha\,|0\rangle + \beta\,|1\rangle) \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) \\ &= \left((-1)^{f(0)}\,\alpha\,|0\rangle + (-1)^{f(1)}\,\beta\,|1\rangle\right) \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right). \end{aligned} \tag{13}$$

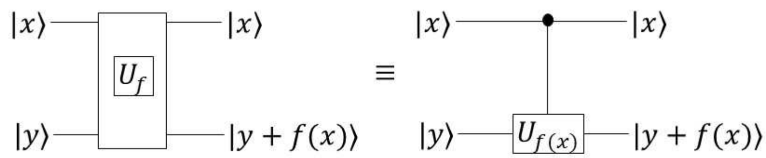

Figure 1: The 2-qubit gate U_f can be thought as a 1-qubit gate $U_{f(x)}$ acting on the second qubit controlled by the first qubit.

Thus we can consider that U_f is a 1-qubit operator c-$U_{f(x)}$ which maps $|y\rangle \mapsto |y \oplus f(x)\rangle$ acting on the second qubit, controlled by the state $|x\rangle$. the corresponding diagram are given below

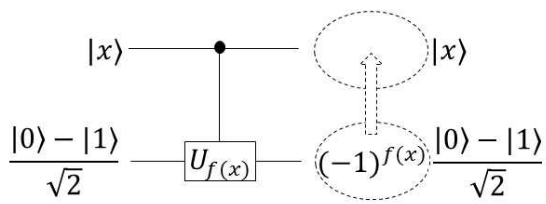

Figure 2: The state $\frac{|0\rangle - |1\rangle}{\sqrt{2}}$ of the target qubit is an eigenstate of $U_{f(x)}$. The eigenvalue $(-1)^{f(x)}$ can be 'kicked back in front of the target qubit.