Dirac notations

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0.1 Dirac Vectors

A notation invented by Dirac which is very useful in quantum mechanics and widely used in quantum computing. Let us consider a particular microscopic system in a particular state. We can represent this state as a particular vector, which we also label, residing in some vector space, where the other elements of the space represent all of the other possible states of the system. Such a space is called a ket space (after Dirac). In the Dirac notation, the symbol identifying a vector is written inside a 'ket' and the state vector is conventionally looks like $|A\rangle$ .Thus, ket vectors differ from conventional vectors in that their magnitudes, or lengths, are physically irrelevant. All the states of the system are in one to one correspondence with all the possible directions of vectors in the ket space, no distinction being made between the directions of the ket vectors $|A\rangle$ and $-|A\rangle$.

The dimensionality of a conventional vector space is defined as the number of independent vectors contained in the space. Likewise, the dimensionality of a ket space is equivalent to the number of independent ket vectors it contains. If there are N independent states, then the possible states of the system are represented as an Ndimensional ket space. Some microscopic systems have a denumerably infinite number of independent states. The possible states of such a system are represented as a ket space whose dimensions are denumerably infinite. Such a space can be treated in more or less the same manner as a finite-dimensional space. Unfortunately, some microscopic systems have a non-denumerably infinite number of independent states. The possible states of such a system are represented as a ket space whose dimensions are nondenumerably infinite. This type of space requires a slightly different treatment to spaces of finite, or denumerably infinite, dimensions.

The states of a general microscopic system can be represented as a complex vector space of infinite dimensions. Such a space is termed a Hilbert space. The Hilbert space of interest for quantum computing will typically has dimensions 2^n, for some positive integer n.

Below are presented the main properties linear algebra object of interest, using the Dirac notation. Since the Hilbert 2^n space used in quantum computing is finite-dimensional we can alternatively present ket vectors as finite column vectors, and represent operators with finite matrices. The standard way to associate column vectors to the ket vectors is as follows

$$\underbrace{|A>}_{ket} = \left.\begin{pmatrix} a \\ b \\ c \\ . \\ . \\ . \end{pmatrix}\right\} \text{column vector} \tag{1}$$

In quantum computing it is convinient to have fixed computational basis. In this basis 2^n ket vectors and corresponding matrix representation using the binary strings of length n are as follows:

$$\Big|\underbrace{00\ldots00}_{n}\Big\rangle = \left.\begin{pmatrix} 1 \\ 0 \\ . \\ . \\ . \\ 0 \\ 0 \end{pmatrix}\right\} 2^n, \qquad \Big|\underbrace{00\ldots01}_{n}\Big\rangle = \left.\begin{pmatrix} 0 \\ 1 \\ . \\ . \\ . \\ 0 \\ 0 \end{pmatrix}\right\} 2^n \quad ...$$

$$\Big|\underbrace{11\ldots10}_{n}\Big\rangle = \begin{pmatrix} 0 \\ 0 \\ . \\ . \\ . \\ 1 \\ 0 \end{pmatrix} \qquad \Big|\underbrace{11\ldots11}_{n}\Big\rangle = \left.\begin{pmatrix} 0 \\ 0 \\ . \\ . \\ . \\ 0 \\ 1 \end{pmatrix}\right\} 2^n$$

For example, $|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

In 2^n Hilbert space any arbitorary ket vector can be written as a weighted sum of above basic vectors using the Dirac notation as well as a single column matrix. However the Dirac notation have some advantages. The Dirac notation are very compact and often preferred because of its elegance. In the Dirac notation we represent a 2^n-dimentional vector by a binary string of lenght n, but the column vector representation would have 2^n components.

To each ket vector $|A\rangle$, there corresponds a dual or adjoint quantity called by Dirac a 'bra' and it conventionally looks like $\langle A|$. It is not a ket– rather it exists in a totally different space and bra vectors are quite different in nature to ket vectors (hence, these vectors are written in mirror image notation, and , so that they can never be confused). Bra space is an example of what mathematicians call a dual vector space -it is dual to the original ket space. There is a one to one correspondence between the elements of the ket space and those of the related bra space. So, for every element of the ket space, there is a corresponding element in the bra space.This is

$$|A\rangle \overset{DC}{\Longleftrightarrow} \langle A| \tag{2}$$

where DC stands for dual correspondence. In terms of matrix representation $\langle A|$ is obtained from $|A\rangle$ by taking the corresponding row matrix, and then taking the complex conjugate of every element. This procedure is known as the 'Hermitean conjugate'. Therefore,

$$\langle A| = \begin{pmatrix} a^* & b^* & c^* & . & . & . \end{pmatrix} \tag{3}$$

Dual to $c\,|A\rangle$ is $c^*\,\langle A|$, where is a complex number. More generally,

$$c_1\,|A\rangle + c_2\,|B\rangle \overset{DC}{\Longleftrightarrow} c_1^*\,\langle A| + c_2^*\,\langle B| \tag{4}$$

Recall from linear algebra the definition of scalar product of a vector $\mathbf{v}$ with vector $\mathbf{w}$ as $\langle \mathbf{v}, \mathbf{w}\rangle$ .The scalar product of these vectors has the following properties:

1. Linearity

$$< \mathbf{v}, \sum_{i=0} \lambda_i \mathbf{w}_i >= \sum_{i=0} \lambda_i < \mathbf{v}, \mathbf{w}_i > .$$

$$< \sum_{i=0} \lambda_i \mathbf{v}_i, \mathbf{w} >= \sum_{i=0} \lambda_i^* < \mathbf{v}_i, \mathbf{w} > .$$

2. Conjugavite-commutativity

$$< \mathbf{v}, \mathbf{w} >=< \mathbf{w}, \mathbf{v} >^*$$

where * denotes complex conjugation.

3. Non-negativity

$$< \mathbf{v}, \mathbf{v} > \quad > 0$$

with equility $< \mathbf{v}, \mathbf{v} >= 0$ if and only if $\mathbf{v} = \mathbf{0}$.

An example of an inner product is the dot product of column vectors and is defined as

$$\mathbf{v}\cdot\mathbf{w} = \begin{pmatrix} v_1 \\ v_2 \\ . \\ . \\ . \\ v_{n-1} \\ v_n \end{pmatrix} \cdot \begin{pmatrix} w_1 \\ w_2 \\ . \\ . \\ . \\ w_{n-1} \\ w_n \end{pmatrix} \tag{5}$$

$$= \begin{pmatrix} v_1^* & v_2^* & . & . & . & v_{n-1}^* & v_n^* \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ . \\ . \\ . \\ w_{n-1} \\ w_n \end{pmatrix} = \sum_{i=1}^{n} v_i^* w_i \tag{6}$$

The generalized scalar product is defined in analogy with the ordinary scalar product that you are familiar with as a combination of a bra and a ket to form a bracket - the generalized scalar product is simply a complex number associated with a pair of kets. The inner product of the vecror $|A\rangle \in \mathcal{H}$ with the vector $|B\rangle \in \mathcal{H}$ is defined as

$$\langle A|\, B\rangle = c \tag{7}$$

where c a complex number. The inner product of $|A\rangle$ with $|B\rangle$ in the matrix representation is computed as the single element of the matrix product representing $\langle A|$ with the column matrix representing $|B\rangle$, which is equivalent to taking the dot product of the column vector associated with $|A\rangle$ with the column vector associated with $|B\rangle$ .The inner product is linear

$$\langle A|\,(c_1\,|B_1\rangle + c_2\,|B_2\rangle) = c_1\langle A\,|B_1\rangle + c_2\langle A\,|B_2\rangle\,. \tag{8}$$

Orthogonality

We say that $|A\rangle$ and $|B\rangle$ are orthogonal if

$$\langle A|\, B\rangle = 0. \tag{9}$$

The Euclidean norm of the vector $|A\rangle$ is denoted as $||A\rangle|$and defined as a square root of the inner product of $|A\rangle$ with itself:

$$|||A\rangle|| = \sqrt{\langle A|\, A\rangle}. \tag{10}$$

The state called normalized if

$$\langle A|\, A\rangle = 1, \tag{11}$$

and two states $|A\rangle$ and $|B\rangle$ are orthonormal if

$$\langle A|\, B\rangle = \delta_{AB}, \tag{12}$$

where δ_{AB} is the Kronecker symbol.

Let's say that a set of 2^n vectors $\{|a_n\rangle\} \subseteq \mathcal{H}$ forms the orthonormal basis for $\mathcal{H}$. Thus, $\langle a_n|\, a_m\rangle = \delta_{nm}$. Every vector $|B\rangle$ from Hilbert space $|B\rangle \in \mathcal{H}$ may be expanded in terms of this orthonormal basis as

$$|B\rangle = \sum_{i=1}^{n} b_i\, |a_i\rangle\,. \tag{13}$$

The coefficients $b_i = \langle B|\, a_i\rangle$ are determined with respect to basis $\{|a_n\rangle\}$ .From this equation it is easy to show that the square of the norm of vector $|B\rangle$ with respect of the orthonormal basis $\{|a_n\rangle\}$ is

$$|||B\rangle||^2 = \sum_{i=1}^{n} |b_i|^2\,. \tag{14}$$

There is very important result that if 2^n vectors $\{|a_n\rangle\} \subseteq \mathcal{H}$ form the orthonormal basis for $\mathcal{H}$ then $\{\langle a_n|\}$ is an orthonormal basis for $\mathcal{H}^*$.

0.2 Operators

Having discussed $|\rangle$ kets , $\langle|$ bras , and $\langle| \ \rangle$ bra-ket pairs, it is now appropriate to study projection operators which are $|\rangle\langle|$ ket-bra products. Recall from linear algebra, a linear operator on a vector space is a linear transformation which maps vectors in $\mathcal{H}$ to vector in $\mathcal{H}$. Let's consider an outer product of ket and bra vectors: $|A\rangle\langle A|$. The meaning of such outer product is that operating on the state vector $|B\rangle$ as $|A\rangle\langle A|\, B\rangle$ this operation reveals the contribution of $|A\rangle$ to $|B\rangle$. In other words this operator projects a vector $|B\rangle$ in $\mathcal{H}$ to the dimensional subspace of $\mathcal{H}$ spanned by $|A\rangle$ .The outer product of the vector $|A\rangle$ with itself $|A\rangle\langle A|$ is called an *orthogonal projector*. More general the outer product $|A\rangle\langle D|$ which applied to $|B\rangle$, acts as follows:

$$(|A\rangle\langle D|)\,|B\rangle = |A\rangle\langle D|\, B\rangle = \langle D|\, B\rangle\,|A\rangle \tag{15}$$

Let's define the operator that are very imprtant for discription of quantum system. Operators are denoted by a hat $\widehat{T}$. The operator $\widehat{T}$ is a linear operator if

$$\widehat{T}(c_c\,|A\rangle + c_2\,|B\rangle) = c_1\widehat{T}\,|A\rangle + c_2\widehat{T}\,|B\rangle\,. \tag{16}$$

The matrix element of an operator is

$$\langle A|\,\widehat{T}\,|B\rangle = \langle A|\,(\widehat{T}\,|B\rangle) = .(\langle A|\,\widehat{T})\,|B\rangle) = c \text{ (a number)}. \tag{17}$$

The expectation value of an operator $\hat{T}$ for a system in state $|A\rangle$ is

$$\left\langle \widehat{T} \right\rangle = \langle A| \widehat{T} |A\rangle = c \text{ (a number)}. \tag{18}$$

Any linear operator difined in $\mathcal{H}$ can be presented as

$$\widehat{T} = . \sum_{i,\ j=1}^{n} T_{ij} |b_i\rangle \langle b_j| . \tag{19}$$

where $T_{ij} = \langle b_i| \hat{T} |b_j\rangle$ are the matrix element and $\{|b_n\rangle\}$ is an orthonormal basis for 2^n vector space $\mathcal{H}$. In other words the linear operator can be presented as all the possible outer products of the pair basis vectors $|b_i\rangle \langle b_j|$. In terms of matrix representation of operator $\hat{T}$, T_{ij} is the matrix element in the i^{th} row and j^{th} column. The action of the operator $\hat{T}$ on the vector $|A\rangle$ is

$$\widehat{T} |A\rangle = . \sum_{i,\ j=1}^{n} T_{ij} |b_i\rangle \langle b_j| A\rangle = \sum_{i,\ j=1}^{n} T_{ij} \langle b_j| A\rangle |b_i\rangle . \tag{20}$$

In quantum physics we have deal with Hermitean operators. Suppose $\hat{T}$ is an operator on $\mathcal{H}$. The hermitean or adjoint operator of $\hat{T}$ denoted $\hat{T}^{\dagger}$ is defined as that linear operator on $\mathcal{H}^*$ that satisfies the following condition:

$$\left(\langle A| \hat{T}^{\dagger} |B\rangle \right)^* = \langle B| \hat{T} |A\rangle \tag{21}$$

When $\hat{T}$ is represented by a matrix the Hermitian conjugate is found by transposing the matrix and then taking the complex conjugate of each matrix element. The operation of taking the Hermitian conjugate of a combination of numbers, states, and operators involves changing $c \to c^*$, $|A\rangle \to \langle A|$, $\langle A| \to |A\rangle$, $\hat{T} \to \hat{T}^{\dagger}$ and reversing the order of all elements. For example,

$$\left(c\hat{P}^{\dagger} \langle A| \hat{T} |B\rangle \langle D| \right)^{\dagger} = c^* |D\rangle \langle B| \hat{T}^{\dagger} |A\rangle \hat{P}. \tag{22}$$

An operator U^{-1} is the inverse operator of U if $U^{-1}U = I$, where I is the identity operator. An operator $\hat{U}$ is unitary if $\hat{U}^{\dagger} = \hat{U}^{-1}$. Note that
$\hat{U}^{\dagger} \hat{U} = \hat{I}$.

Observables in quantum mechanics are represented by Hermitian operators which satisfy $\hat{T} = \hat{T}^{\dagger}$.The expectation value of a Hermitian operator is real:

$$\left\langle \widehat{T} \right\rangle^* = \left(\langle A| \widehat{T} |A\rangle \right)^* = \left(\langle A| \widehat{T}^\dagger |A\rangle \right) = \langle A| \widehat{T} |A\rangle = \left\langle \widehat{T} \right\rangle \tag{23}$$

A vector $|A\rangle$ is an eigenvector of an operator $\hat{T}$ if

$$\widehat{T} |A\rangle = c |A\rangle . \tag{24}$$

In Equation ( ) c is a constant and is called the eigenvalue of T corresponding to eigenvector $|A\rangle$ .It is easy to prove that the eigenvalues of a Hermitean operator are real.

$$\widehat{T} |A\rangle = c |A\rangle . \tag{25}$$

$$\widehat{T}^\dagger |A\rangle = c^* |A\rangle \tag{26}$$

Therefore, we have that $(\widehat{T}^\dagger - \hat{T})^\dagger |A\rangle = (c - c^*) |A\rangle = 0$. thus, $c = c^*$.

Another famous theorem: eigenvectors of the same Hermitian operator having different eigenvalues are automatically orthogonal. Denote the eigenstates of a Hermitian operator by $|n\rangle$. Let's show that states corresponding to different eigenvalues are orthogonal.

We assume the states are normalized so that $\langle m |n\rangle = \delta_{mn}$. Suppose that $\widehat{A} |n\rangle = a_n |n\rangle$. To prove orthogonality we calculate

$$\langle m| \widehat{A} |n\rangle = \langle m| a_n |n\rangle = a_n \langle m| n\rangle \tag{27}$$

and

$$\langle m| \widehat{A} |n\rangle = (\hat{A}^\dagger |m\rangle)^\dagger |n\rangle = (a_m |m\rangle)^\dagger |n\rangle = a_m^* \langle m| n\rangle = a_m \langle m| n\rangle. \tag{28}$$

Thus,

$$(a_n - a_m) \langle m| n\rangle = 0 \tag{29}$$

so $\langle m| n\rangle = 0$ if $m = n$.

The eigenstates of a Hermitian operator form a complete set. Therefore for an arbitrary vector $|A\rangle$

$$|A\rangle = \sum_{n=1}^{n} a_n |n\rangle , \tag{30}$$

where

$$a_n = \langle n |A\rangle = \langle n| \sum_{m=1}^{n} a_n |m\rangle = \sum_{m=1}^{n} a_n \langle n |m\rangle = \sum_{m=1}^{n} a_n \delta_{nm} = a_n. \tag{31}$$

$$|A\rangle = .\sum_{n=1}^{n} a_n |n\rangle = \sum_{n=1}^{n} \langle n |A\rangle |n\rangle = \sum_{n=1}^{n} |n\rangle \langle n |A\rangle = \left(\sum_{n=1}^{n} |n\rangle \langle n| \right) |A\rangle \tag{32}$$

Since this is true for arbitrary $|A\rangle$ we can write the identity operator as

$$\hat{I} = \sum_{n=1}^{n} |n\rangle \langle n| \,. \tag{33}$$

A component of $|A\rangle$ can be found by operating with the projection operator $\hat{P}_n = |n\rangle \langle n|$. We have

$$\hat{P}_n |A\rangle = .|n\rangle \langle n| \sum_{m=1}^{n} a_m |m\rangle = |n\rangle \sum_{m=1}^{n} a_m \langle n |m\rangle = a_n |n\rangle \,. \tag{34}$$

The projection operator is idempotent:

$$\left(\hat{P}_n\right)^2 = .\hat{P}_n \hat{P}_n = (|n\rangle \langle n|)(|n\rangle \langle n|) = |n\rangle \left(\langle n| \, n\rangle\right) \langle n| = |n\rangle \langle n| = \hat{P}_n. \tag{35}$$

The inner product of two states can be expressed in terms of the coefficients of their decomposition. We write $|A\rangle = .\sum_{n=1}^{n} a_n |n\rangle$ and $|B\rangle = .\sum_{n=1}^{n} b_n |n\rangle$. Then

$$\langle B |A\rangle = .\sum_{m=1}^{n} b_m^* \langle m| \sum_{n=1}^{n} a_n |n\rangle = \sum_{m=1}^{n} \sum_{n=1}^{n} b_m^* a_n \langle m |n\rangle = \sum_{m=1}^{n} \sum_{n=1}^{n} b_m^* a_n \delta_{mn} = \sum_{n=1}^{n} b_n^* a_n \tag{36}$$

1 Examples

The example given in the text shows that:

$$\sqrt{\frac{2}{3}}|01> + \frac{i}{\sqrt{3}}|11> = \sqrt{\frac{2}{3}}|0> \bigotimes |1> + \frac{i}{\sqrt{3}}|1> \bigotimes |1>$$

It appears that the 2 single digit kets can be combined together to be rewritten as a single 2-digit ket.

$$|n_1> \bigotimes |n_2> = |n_1 n_2>$$

And the two digit kets (N=2) can be represented in matrix form as 4 dimensional column vectors ($2^{N=2} = 2^2 = 4$).

Example goes over the sum of two 4-dimensional vectors:

$$\sqrt{\frac{2}{3}}\,|01\rangle + \frac{i}{\sqrt{3}}\,|11\rangle = \sqrt{\frac{2}{3}}\begin{pmatrix}0\\1\\0\\0\end{pmatrix} + \frac{i}{\sqrt{3}}\begin{pmatrix}0\\0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\\sqrt{\frac{2}{3}}\\0\\\frac{i}{\sqrt{3}}\end{pmatrix}$$

The evaluation of the dot/inner product of column vectors **w** and **v** follows:

$$\mathbf{v}\cdot\mathbf{w} = \begin{pmatrix}v_1\\v_2\\.\\.\\.\\v_{n-1}\\v_n\end{pmatrix}\cdot\begin{pmatrix}w_1\\w_2\\.\\.\\.\\w_{n-1}\\w_n\end{pmatrix} \tag{37}$$

$$= \begin{pmatrix}v_1^* & v_2^* & . & . & . & v_{n-1}^* & v_n^*\end{pmatrix}\begin{pmatrix}w_1\\w_2\\.\\.\\.\\w_{n-1}\\w_n\end{pmatrix} \tag{38}$$

$$= \sum_{i=1}^{n} v_i^* w_i \tag{39}$$

$$\tag{40}$$

The example given in the text

$$\begin{pmatrix} 0 \\ \sqrt{\frac{2}{3}} \\ 0 \\ \frac{i}{\sqrt{3}} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ \sqrt{\frac{1}{2}} \\ \sqrt{\frac{1}{2}} \end{pmatrix} = \begin{pmatrix} 0 & \sqrt{\frac{2}{3}} & 0 & \frac{-i}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \sqrt{\frac{1}{2}} \\ \sqrt{\frac{1}{2}} \end{pmatrix} \tag{41}$$

$$= 0 \cdot 0 + \sqrt{\frac{2}{3}} \cdot 0 + 0 \cdot \sqrt{\frac{1}{2}} + \frac{-i}{\sqrt{3}} \cdot \sqrt{\frac{1}{2}} \tag{42}$$

$$= \frac{-i}{\sqrt{3}} \cdot \sqrt{\frac{1}{2}} \tag{43}$$

$$\begin{pmatrix} 0 \\ \sqrt{\frac{2}{3}} \\ 0 \\ \frac{i}{\sqrt{3}} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ \sqrt{\frac{1}{2}} \\ \sqrt{\frac{1}{2}} \end{pmatrix} = \frac{-i}{\sqrt{6}} \tag{44}$$

When performing the inner products of vectors of Hilbert space
($< \chi| \in \mathcal{H}^* : |\psi >$ and $|\psi > \in \mathcal{H}$):

$$< \chi| : |\psi > \rightarrow < \chi|\psi > \in \mathbf{C}$$

With the $|\psi > \in \mathcal{H}$, the dual of which is, $< \psi| \in \mathcal{H}^*$: The norm is defined as:

$$||\psi > || = \sqrt{< \psi|\psi >}$$

Now, let's say that within $\mathcal{H}$ space, we were able to find orthonormal vectors b_n such that they span all space of the $\mathcal{H}$.

$$< b_n|b_m > = \delta_{n,m}$$

where

$$\forall b_m, b_n \in B$$

So, basically, $|\psi >$ can be rewritten with orthonormal bases.

$$|\psi > = \sum_{b_n \in B} \psi_n |b_n >$$

with $\psi_n = < b_n|\psi >$

The example given in the previous page can be performed in Dirac notation. With more elegance, it appears to be more easily done:

$$
\begin{aligned}
<\psi|\phi> &= \left(\sqrt{\frac{2}{3}}\langle 01| + \frac{-i}{\sqrt{3}}\langle 11|\right)\left(\sqrt{\frac{1}{2}}|10\rangle + \sqrt{\frac{1}{2}}|11\rangle\right) && (45)\\
&= (\sqrt{\frac{2}{3}})(\sqrt{\frac{1}{2}})\underbrace{<01|10>}_{0} + (\sqrt{\frac{2}{3}})(\sqrt{\frac{1}{2}})\underbrace{<01|11>}_{0} && (46)\\
&\quad + (\frac{-i}{\sqrt{3}})(\sqrt{\frac{1}{2}})\underbrace{<11|10>}_{0} + (\frac{-i}{\sqrt{3}})(\sqrt{\frac{1}{2}})\underbrace{<11|11>}_{1} && (47)\\
<\psi|\phi> &= \frac{-i}{\sqrt{6}} && (48)
\end{aligned}
$$

Hadamard basis $|+> and|->$, defined as:

$$
\begin{aligned}
|+> &= \frac{1}{\sqrt{2}}(|0> + |1>) && (49)\\
|-> &= \frac{1}{\sqrt{2}}(|0> - |1>) && (50)
\end{aligned}
$$

The inner product of the two:

$$
\begin{aligned}
<+|-> &= \frac{1}{\sqrt{2}}(<0| + <1|)\frac{1}{\sqrt{2}}(|0> - |1>) && (51)\\
&= \frac{1}{2}\left(\underbrace{<0|0>}_{1} - \underbrace{<0|1>}_{0} + \underbrace{<1|0>}_{0} - \underbrace{<1|1>}_{1}\right) && (52)\\
&= 0 && (53)
\end{aligned}
$$

The above states that the two Hadamard bases are orthogonal from each other.

Let's try evaluating this in column matrix form to see if they agree with the above:

$$
\begin{aligned}
<+|-> &= \frac{1}{2}(<0| + <1|)(|0> - |1>) && (54)\\
&= \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} && (55)\\
&= \frac{1}{2}\begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} && (56)\\
&= \frac{1}{2}(1 \cdot 1 + 1 \cdot -1) && (57)\\
<+|-> &= 0 && (58)
\end{aligned}
$$

The norm of the 1st Hadarmard basis $|+>$:

$$\begin{aligned}
|||+>||^2 &= <+|+> && (59)\\
&= \frac{1}{2}(<0|+<1|)(|0>+|1>) && (60)\\
&= \frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix}\cdot\begin{pmatrix}1\\1\end{pmatrix} && (61)\\
&= \frac{1}{2}\begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} && (62)\\
&= \frac{1}{2}(1\cdot 1+1\cdot 1) && (63)\\
|||+>||^2 &= 1 && (64)
\end{aligned}$$

The norm of the 2nd Hadarmard basis $|->$:

$$\begin{aligned}
|||->||^2 &= <-|-> && (65)\\
&= \frac{1}{2}(<0|-<1|)(|0>-|1>) && (66)\\
&= \frac{1}{2}\begin{pmatrix}1\\-1\end{pmatrix}\cdot\begin{pmatrix}1\\-1\end{pmatrix} && (67)\\
&= \frac{1}{2}\begin{pmatrix}1 & -1\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix} && (68)\\
&= \frac{1}{2}(1\cdot 1+-1\cdot -1) && (69)\\
|||->||^2 &= 1 && (70)
\end{aligned}$$

Operators can be thought of outer products which transforms $\mathcal{H}\to\mathcal{H}$:

$$\begin{aligned}
(|\psi><\phi|)|\gamma> &= |\psi>(<\phi|\gamma>) && (71)\\
&= (<\phi|\gamma>)|\psi> && (72)
\end{aligned}$$

In this case, the input vector $|\gamma>\in\mathcal{H}$ and after operation, the result vector, $|\psi>\in\mathcal{H}$

Linear operator T can be rewritten with the kets and bras $|b_n>\in B$:

$$T = \sum_{b_n,b_m\in B} T_{n,m}|b_n><b_m| \qquad (73)$$

alongside, T operator can be written as in matrix form:

$$T = \begin{pmatrix} T_{11} & T_{12} & . & . & T_{1,N-1} & T_{1N} \\ T_{21} & T_{22} & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ T_{N1} & T_{N2} & . & . & T_{N,N-1} & T_{NN} \\ \end{pmatrix}$$

When operator T is applied onto an arbitrary vector $|\psi>$:

$$\begin{aligned} T|\psi> &= \sum_{b_n, b_m \in B} T_{n,m}|b_n><b_m||\psi> && (74) \\ &= \sum_{b_n, b_m \in B} T_{n,m}(<b_m|\psi>)|b_n> && (75) \\ &= \sum_{b_n, b_m \in B} T_{n,m}\psi_m|b_n> && (76) \end{aligned}$$

where ψ_m is defined as $<b_m|\psi>$

So, say that we want to now solve for the elements of some operator Z, but only know what Z does to $|0>$ and $|1>$.Knowing that

$Z|0>=|0>$ and $Z|1>=-|1>$

We can write Z as a 2x2 matrix

$$Z = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix}$$

Applying it to $|0>$, we can solve for the elements Z_{11} and Z_{12}:

$$\begin{aligned} Z|0> &= |0> && (77) \\ \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} && (78) \\ Z_{11} &= 1 && (79) \\ Z_{21} &= 0 && (80) \\ Z &\rightarrow \begin{pmatrix} 1 & Z_{12} \\ 0 & Z_{22} \end{pmatrix} && (81) \end{aligned}$$

We have solved for the elements Z_{11} and Z_{12}. Now, let's apply Z operator to $|1>$ to solve for the rest of the Z matrix.

$$
\begin{aligned}
Z|1> &= -|1> && (82)\\
\begin{pmatrix} 1 & Z_{12} \\ 0 & Z_{22} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} && (83)\\
Z_{12} &= 0 && (84)\\
Z_{22} &= -1 && (85)\\
Z &\rightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} && (86)
\end{aligned}
$$

The Identity operator can be defined as the following using a set of orthogonal vectors $|b_n>$

$$\mathbf{1} = \sum_{b_n \in B} |b_n >< b_n|$$

A unitary operator U is defined such that: $U^\dagger = U^{-1}$ where U^{-1} is the inverse of U.
If $|\psi>$ is eigenvector of operator T, then we have:

$$T|\psi >= c|\psi >$$

then

$$c \in \mathbf{C}$$

where c would be the called the eigenvalue of T.
If the T operator is Hermitian, $T = T^\dagger$, then

$$T|\psi >= c|\psi >$$

then

$$c \in \mathbf{R}$$

The trace of an operator is defined as:

$$Tr(A) = \sum_{b_n} < b_n|A|b_n >$$

where the choice of the orthonormal basis is irrelevant.

Commutativity of Operators: A,B commute if: AB=BA or AB-BA=0. Or in short, [A,B]=0.
Spectral theorem says that if an operator T is a matrix operator, there is a unitary operator P such that: $T = P\Lambda P^\dagger$ where Λ is a diagonal matrix

$$X|0>=|1>X|1>=|0>$$

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{87}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix} \tag{88}$$

where

$$P = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix}$$

The eigenvalues of X are 1 and -1 and the eigenvectors of X are

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix}$$

respectively. These look like the Hadamard basis:

$$|+>=\frac{1}{\sqrt{2}}|0>+\frac{1}{\sqrt{2}}|1>, |->=\frac{1}{\sqrt{2}}|0>-\frac{1}{\sqrt{2}}|1>$$

Every normal operator can also be written in diagonal form:

$$T = \sum_i T_i |T_i><T_i|$$

where $|T_i><T_i|$ is a projector and $<T_i|T_j>=\delta_{i,j}$

$$\begin{aligned} f(T) &= \sum_m a_m T^m & (89) \\ &= \sum_m a_m \left(\sum_i T_i |T_i><T_i| \right)^m & (90) \\ &= \sum_m a_m \sum_i T_i^m |T_i><T_i| & (91) \\ &= \sum_i \left(\sum_m a_m T_i^m \right) |T_i><T_i| & (92) \\ &= \sum_i f(T_i) |T_i><T_i| & (93) \\ & & (94) \end{aligned}$$

Tensor products were also reviewed in the chapter.

$$A \bigotimes B = \begin{pmatrix} A_{11}B_{11} & ... & A_{11}B_{1q} & ... & ... & A_{1n}B_{11} & A_{1n}B_{1q} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ A_{11}B_{p1} & ... & A_{11}B_{pq} & ... & ... & A_{1n}B_{p1} & A_{1n}B_{pq} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ A_{m1}B_{11} & ... & A_{m1}B_{1q} & ... & ... & A_{mn}B_{11} & A_{mn}B_{1q} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ A_{m1}B_{p1} & ... & A_{m1}B_{pq} & ... & ... & A_{mn}B_{p1} & A_{mn}B_{pq} \end{pmatrix} \tag{95}$$

A more compact way of writing this is:

$$A \bigotimes B = \begin{pmatrix} A_{11}[B] & A_{12}[B] & ... & A_{1n}[B] \\ A_{21}[B] & A_{22}[B] & ... & A_{2n}[B] \\ . & . & . & . \\ A_{m1}[B] & A_{m2}[B] & ... & A_{mn}[B] \end{pmatrix}$$

where

$$A_{ij}[B] = \begin{pmatrix} A_{ij}B_{11} & A_{ij}B_{12} & ... & A_{ij}B_{1q} \\ A_{ij}B_{21} & A_{ij}B_{22} & ... & A_{ij}B_{2q} \\ . & . & . & . \\ A_{ij}B_{p1} & A_{ij}B_{p2} & ... & A_{ij}B_{pq} \end{pmatrix} \tag{96}$$

For a tensor multiplication, $(\alpha_0|0> + \alpha_1|1>) \bigotimes (\beta_0|0> + \beta_1|1>)$ can be represented in matrix form.

$$\begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} \bigotimes \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} \alpha_0\beta_0 \\ \alpha_0\beta_1 \\ \alpha_1\beta_0 \\ \alpha_1\beta_1 \end{pmatrix}$$

The Schmidt Decomposition Theorem states that if $|\psi>$ is vector in tensor product space $(\mathcal{H}_A \bigotimes \mathcal{H}_B)$, then you can represent it as a sum of sets of tensor products of orthonormal bases $|\psi_i^A> \in \mathcal{H}_A$ and $|\psi_j^B> \in \mathcal{H}_B$, such as the following:

$$|\psi> = \sum_i \sqrt{p_i} |\phi_i^A> |\phi_i^B>$$

Another way to state this is that we can always find a set of orthonormal bases from each space $(\mathcal{H})$ so that the cross terms vanish. In this case, we should be able to find a set of orthonormal bases from $\mathcal{H}_A$ and another set of orthonormal bases from $\mathcal{H}_B$ so that we would only be summing up the without doubly-nesting the summation.