Chapter 2.
Elements of Linear Algebra

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0.1 Elements of Linear Algebra

There are mathematical structures that are based on complex numbers. All the vector spaces we have studied thus far in the text are real vector spaces since the scalars are real numbers. A complex vector spaces one in which the scalars are complex numbers z. The primary example of a complex vector space is the set of vectors of a fixed length with complex entries. Let's consider one concrete example. We denote set of $\mathrm{C}^4 = \mathrm{C}\times\mathrm{C}\times\mathrm{C}$, which reminds that each vector is an ordered list of three complex numbers. The general and typical element of C^4 which we call vector V looks like this

$$V = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} \text{ or as an example } V = \begin{bmatrix} 5 + i4 \\ 6 - i2 \\ 5.4 + i3 \\ 6.7 - i4.5 \end{bmatrix}. \tag{1}$$

The jth element of this vector is $V[j]$.For example, in (1) $V[3] = 5.4 + i3$. We also define a distinguished vector call zero

$$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{2}$$

There is exist in C^4 the other vector that called inverse vector such that

$$V - V = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} + \begin{bmatrix} -z_1 \\ -z_2 \\ -z_3 \\ -z_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \mathbf{0} \tag{3}$$

Algebra in vector space. In Let's define the addition and multiplication such vectors. For given V and W vectors in C^n . Addition $V + W = W + V$. Formally, this operation amount to $(V + W)[j] = V[j] + W[j]$.

Scalar multiplication $c \cdot V$, where c is a scalar, formally is understood $(c \cdot V)[j] = c \times V[j]$. The scalar multiplication satisfies the following properties for all $c, c_1, c_2 \in$C and for all $V, W \in$C^n :

$$\begin{gathered} 1 \cdot V = V \\ c_1 \cdot (c_2 \cdot V) = (c_1 \times c_2) \cdot V \\ c \cdot (V + W) = c \cdot V + c \cdot W) \\ (c_1 + c_2) \cdot V = c_1 \cdot V + c_2 \cdot V \end{gathered} \tag{4}$$

The formal definition of complex vector space is the set whose elements are called vectors with a distinguished element zero vector and three operations: addition, negation, scalar multiplication. For all $V, W, Y \in$V and $c, c_1, c_2 \in$C these operations and zero vector must satisfy the following properties:

(1) Commutativity of addition: $V + W = W + V$.

(2) Associativity of addition: $(V + W) + Y = V + (W + Y)$.

(3) Zero is an additive identity: $V + \mathbf{0} = \mathbf{0} + V = V$.

(4) Every vector has an inverse: $V + (-V) = \mathbf{0} = (-V) + V$.

(5) Scalar multiplication has a unit $1 \cdot V = V$.

(6) Scalar multiplication respects complex multiplication: $c_1 \cdot (c_2 \cdot V) = (c_1 \times c_2) \cdot V$.

(7) Scalar multiplication distributes over addition: $c \cdot (V + W) = c \cdot V + c \cdot W)$.

(8) Scalar multiplication distributes over complex addition: $(c_1 + c_2) \cdot V = c_1 \cdot V + c_2 \cdot V$.

There is a real vector space $\mathbf{R}$, which is like a complex vector space where the scalar multiplication to be defined for scalars in $\mathbf{R} \in \mathbf{C}$.

Now let us consider the vector in C$^{n \times n}$ vector space. The given element $A \in$C$^{n \times n}$ is represented by $n \times n$ matrix with complex entry in kth row and jth column as $A[k, j]$ or $c_{k,j}$. The general form of this element is

$$V = \begin{bmatrix} c_{0,0} & c_{0,1} & \dots & c_{0,n-1} \\ c_{1,0} & c_{1,1} & \dots & c_{1,n-1} \\ \dots & \dots & \dots & \dots \\ c_{n-1,0} & c_{n-1,1} & \dots & c_{n-1,n-1} \end{bmatrix} \tag{5}$$

One can say that the vector considered above are the special case of the matrix and represent just a one column of the matrix.

Below we consider the operation that one can perform on the matrix $A \in$C$^{n \times n}$: i) Transpoce, ii) Congugate, iii) Dagger operations. The transpose matrix of A, denote as A^T is defined as $A^T[k, j] = A[j, k]$. In other words the element at row k column j in the original is placed at row j column k of the transpose. For example let consider 3×3 matrix;

$$A = \begin{bmatrix} 5 + i4 & 5i & 3 - i4 \\ -i2 & 5 & 2 + 3i \\ 4 + i3 & 1 - i3 & 6 - i2 \end{bmatrix}, \ A^T = \begin{bmatrix} 5 + i4 & 5i & 3 - i4 \\ -i2 & 5 & 2 + 3i \\ 4 + i3 & 1 - i3 & 6 - i2 \end{bmatrix}^T = \begin{bmatrix} 5 + i4 & -i2 & 4 + i3 \\ 5i & 5 & 1 - i3 \\ 3 - i4 & 2 + 3i & 6 - i2 \end{bmatrix}. \tag{6}$$

The conjugate of the matrix A, denoted as $\bar{A}$, is the matrix where each elements is the complex congugate of the corresponding elements of the original matrix A.Therefore, $\bar{A}[k,j] = A\overline{[k,j]}$. For example, for the matrix A given in (6) we have

$$\bar{A} = \overline{\begin{bmatrix} 5+i4 & 5i & 3-i4 \\ -i2 & 5 & 2+3i \\ 4+i3 & 1-i3 & 6-i2 \end{bmatrix}} = \begin{bmatrix} 5-i4 & -5i & 3+i4 \\ i2 & 5 & 2-3i \\ 4-i3 & 1+i3 & 6+i2 \end{bmatrix}. \quad (7)$$

The transpose operation and conjugate operation can be combined to form the dagger or adjoint operation. The dagger operation is denoted as $A^\dagger$ and is defined as $A^\dagger = (\overline{A^T})$. Thus, $A^\dagger[k,j] = A\overline{[j,k]}$. For example the dagger operation for the matrix A given in (6) gives

$$A^\dagger = \begin{bmatrix} 5+i4 & 5i & 3-i4 \\ -i2 & 5 & 2+3i \\ 4+i3 & 1-i3 & 6-i2 \end{bmatrix}^\dagger = \overline{\begin{bmatrix} 5+i4 & -i2 & 4+i3 \\ 5i & 5 & 1-i3 \\ 3-i4 & 2+3i & 6-i2 \end{bmatrix}} = \begin{bmatrix} 5-i4 & i2 & 4-i3 \\ -5i & 5 & 1+i3 \\ 3+i4 & 2-3i & 6+i2 \end{bmatrix}. \quad (8)$$

Above defined operations satisfy the following properties for all $c \in \mathrm{C}$ and all $A, B \in \mathrm{C}^{n\times n}$:

(1) Transpose is idempotent $(A^T)^T = A$.
(2) Transpose respects addition $(A+B)^T = A^T + B^T$.
(3) Transpose respects scalar multiplication $(c \cdot A)^T = c \cdot A^T$.
(4) Conjugate is idempotent $\bar{\bar{A}} = A$.
(5) Conjugate respects addition $\overline{(A+B)} = \bar{A} + \bar{B}$.
(6) Conjugate respects scalar multiplication $\overline{(c \cdot A)} = \bar{c} \cdot \bar{A}$.
(7) Dagger is idempotent $(A^\dagger)^\dagger = A$.
(8) Dagger respects addition $(A+B)^\dagger = A^\dagger + B^\dagger$.
(9) Dagger respects scalar multiplication $(c \cdot A)^\dagger = \bar{c} \cdot A^\dagger$.

0.2 Addition and multiplication of matrixes

Consider two matrices A and B

$$A = \begin{bmatrix} a_{0,0} & a_{0,1} & \dots & a_{0,n-1} \\ a_{1,0} & a_{1,1} & \dots & a_{1,n-1} \\ \dots & \dots & \dots & \dots \\ a_{n-1,0} & a_{n-1,1} & \dots & a_{n-1,n-1} \end{bmatrix} \text{ and } B = \begin{bmatrix} b_{0,0} & b_{0,1} & \dots & b_{0,n-1} \\ b_{1,0} & b_{1,1} & \dots & b_{1,n-1} \\ \dots & \dots & \dots & \dots \\ b_{n-1,0} & b_{n-1,1} & \dots & b_{n-1,n-1} \end{bmatrix} \quad (9)$$

The sum of these matrices is defined as $C[k,j] = (A+B)[k,j] = A[k,j] + B[k,j]$:

$$C = A{+}B = \begin{bmatrix} a_{0,0}+b_{0,0} & a_{0,1}+b_{0,1} & ... & a_{0,n-1}+b_{0,n-1} \\ a_{1,0}+b_{1,0} & a_{1,1}+b_{1,1} & ... & a_{1,n-1}+b_{1,n-1} \\ ... & ... & ... & ... \\ a_{n-1,0}+b_{n-1,0} & a_{n-1,1}+b_{n-1,1} & ... & a_{n-1,n-1}+b_{n-1,n-1} \end{bmatrix} \tag{10}$$

For example,

$$A{+}B = \begin{bmatrix} 5+i4 & 5i & 3-i4 \\ -i2 & 5 & 2+i3 \\ 4+i3 & 1-i3 & 6-i2 \end{bmatrix} + \begin{bmatrix} 2-i3 & -i2 & 1+i3 \\ 1+i5 & 2 & 1-i3 \\ 1-i4 & 3+2i & 2-i3 \end{bmatrix} = \begin{bmatrix} 7+i & i3 & 4-i \\ 1+i3 & 7 & 3 \\ 5-i & 4-i & 8-i5 \end{bmatrix}. \tag{11}$$

To find the element of $(A{\star}B)[k,j]$ of matrix product denoted $A{\star}B$ one should multiply each element of the k row of the matrix A with the corresponding elements of the j column of the matrix B and sum the results. Formally we construct $A \star B \in \mathrm{C}^{n\times n}$ as $(A \star B)[k,j] = \sum_{l=0}^{n-1}(A[k,l] \times B[l,j])$. To introduce the multiplication of two square matrices let us consider multiplication of two 3×3 A and B matrices:

$$A{\star}B = \begin{bmatrix} 5+i4 & 5i & 3-i4 \\ -i2 & 5 & 2+i3 \\ 4+i3 & 1-i3 & 6-i2 \end{bmatrix} \star \begin{bmatrix} 2-i3 & -i2 & 1+i3 \\ 1+i5 & 2 & 1-i3 \\ 1-i4 & 3+2i & 2-i3 \end{bmatrix} = \begin{bmatrix} 16-i18 & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix}. \tag{12}$$

The identity matrix is defined as $n \times n$ matrix with unity elements on the main diagonal:

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & ... & 0 \\ ... & ... & 1 & 0 \\ 0 & 0 & ... & 1 \end{bmatrix} \tag{13}$$

and it plays a role of a unit matrix multiplication. Therefore, $I \star A = A \star I = A$. Matrix multiplication satisfies the following properties. For all $A, B, C \in \mathrm{C}^{n\times n}$

(1) Matrix multiplication is associative $(A \star B) \star C = A \star (B \star C)$

(2) Matrix multiplication distributes over addition
$A \star (B + C) = (A \star B) + (A \star C)$; $(A + B) \star C = (A \star C) + (B \star C)$.

(3) Matrix multiplication respect scalar multiplication
$c \cdot (A \star B) = (c \cdot A) \star B) = A \star (c \cdot B)$.

(4) Matrix multiplication relates to the transpose operation
$(A \star B)^T = B^T \star A^T$.

(5) Matrix multiplication respect conjugate operation $(\bar{A} \star \bar{B}) = \bar{A} \star \bar{B}$

(6) Matrix multiplication relates to the dagger operation $(A{\star}B)^\dagger = B^\dagger {\star} A^\dagger$.

(7) Commutativity is not a main property of the matrix multiplication.

Performing the exercises given at the end of this Chapters you can justify all above mentioned properties.

0.3 Basis

We can introduce canonical basis or the standard basis which allows to present any complex vector or matrices in vector space. For example In R^3 the basis is the following

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 \ , & 1 \ , & 0 \\ 0 & 0 & 1 \end{bmatrix} , \tag{14}$$

while the canonical basis for the vector space consists of matrices is

$$E_{k,j} = \begin{bmatrix} 0 & 0 & ... & 0 \\ 0 & 0 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & 1 & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & 0 \end{bmatrix} , \tag{15}$$

where $E_{k,j}$ has 1 in row k column j and 0's everywhere else. Therefore any $n \times n$ matrix can be written as $A = \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} A[k,j] E_{k,j}$.

Let's define the inner product with two vectors in R^3 by considering the following example:

$$\left\langle \begin{bmatrix} 5 \\ 3 \\ -7 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 2 \\ 0 \end{bmatrix} \right\rangle = \begin{bmatrix} 5 & 3 & -7 \end{bmatrix}^T \star \begin{bmatrix} 6 \\ 2 \\ 0 \end{bmatrix} = (5\times6)+(3\times2)+(-7\times0) = 36 \ . \tag{16}$$

In general for two vectors V_1 and V_2 the inner product is $\langle V_1, V_2 \rangle = V_1^T \star V_2$, and satisfies the following conditions:

(1) Nonnegativity $\langle V, V \rangle > 0$, and $\langle V, V \rangle = 0$ if only $V = 0$.

(2) Respect addition $\langle V_1 + V_2, V_3 \rangle = \langle V_1, V_3 \rangle + \langle V_2, V_3 \rangle$ or $\langle V_1, V_2 + V_3 \rangle = \langle V_1, V_2 \rangle + \langle V_1, V_3 \rangle$.

(3) Respects scalar multiplication $\langle c \cdot V_1, V_2 \rangle = c \times \langle V_1, V_2 \rangle$ or $\langle V_1, c \cdot V_2 \rangle = \bar{c} \times \langle V_1, V_2 \rangle$.

(4) Skew symmetric $\langle V_1, V_2 \rangle = \langle \bar{V_2}, \bar{V_1} \rangle$.

The above relations are valid for any $V_1, V_2, V_3 \in V$ and $c \in$C.

A matrix A is called symmetric if $A^T = A$. The matrix is called Hermitian if $A^\dagger = A$.

Show that matrix $\begin{bmatrix} 7 & 6 + i5 \\ 6 - i5 & -3 \end{bmatrix}$ is hermition.

The other important class of matrix is unitary. The matrix is unitary if $U \star U^\dagger = U^\dagger \star U = I$

Show that matrix $\begin{bmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{bmatrix}$ is unitary.

In quantum mechanics if V_1 describes one system and V_2 describes another system, then their tensor product describes both systems as one. Therefore the tensor product is important operation for description of the quantum system. Let us introduce the tensor product of vector and matrices through the examples. The notation for the tensor product is $\otimes$. The tensor product of two vector is defined as follows:

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} \otimes \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} a_0 \cdot & \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} \\ a_1 \cdot & \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} \\ a_2 \cdot & \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} a_0 b_0 \\ a_0 b_1 \\ a_0 b_2 \\ a_1 b_0 \\ a_1 b_1 \\ a_1 b_2 \\ a_2 b_0 \\ a_2 b_1 \\ a_2 b_2 \end{bmatrix} \tag{17}$$

We need also to know how to determine the tensor product of two matrices. For simplicity let's consider the tensor product of two 2×2 A and B matrices

$$\begin{aligned} A \otimes B &= \begin{bmatrix} a_{0,0} & a_{0,1} \\ a_{1,0} & a_{1,1} \end{bmatrix} \otimes \begin{bmatrix} b_{0,0} & b_{0,1} \\ b_{1,0} & b_{1,1} \end{bmatrix} = \begin{bmatrix} a_{0,0} \cdot \begin{bmatrix} b_{0,0} & b_{0,1} \\ b_{1,0} & b_{1,1} \end{bmatrix} & a_{0,1} \cdot \begin{bmatrix} b_{0,0} & b_{0,1} \\ b_{1,0} & b_{1,1} \end{bmatrix} \\ a_{1,0} \cdot \begin{bmatrix} b_{0,0} & b_{0,1} \\ b_{1,0} & b_{1,1} \end{bmatrix} & a_{1,1} \cdot \begin{bmatrix} b_{0,0} & b_{0,1} \\ b_{1,0} & b_{1,1} \end{bmatrix} \end{bmatrix} = \\ &= \begin{bmatrix} a_{0,0} \times b_{0,0} & a_{0,0} \times b_{0,1} & a_{0,1} \times b_{0,0} & a_{0,1} \times b_{0,1} \\ a_{0,0} \times b_{1,0} & a_{0,0} \times b_{1,1} & a_{0,1} \times b_{1,0} & a_{0,1} \times b_{1,1} \\ a_{1,0} \times b_{0,0} & a_{1,0} \times b_{0,1} & a_{1,1} \times b_{0,0} & a_{1,1} \times b_{0,1} \\ a_{1,0} \times b_{1,0} & a_{1,0} \times b_{1,1} & a_{1,1} \times b_{1,0} & a_{1,1} \times b_{1,1} \end{bmatrix}. \end{aligned} \tag{18}$$

It can be seen that the tensor product $A \otimes B$ is the matrix that has every element of A matrix, scalar multiplied with the entire matrix B.

1 Homework Exercises

Given two matrices

$$A = \begin{bmatrix} 2+i4 & 5i & 3-i4 \\ 1-i2 & 3 & 1+i3 \\ 2+i3 & 5-i3 & 4-i2 \end{bmatrix} \quad \text{and } B = \begin{bmatrix} 2-i3 & -i & 1+i3 \\ 2+i3 & 2i & 1-i3 \\ 1-i4 & 5+2i & 1-i3 \end{bmatrix}$$

a) Find
(1) $A + B$
(2) $A \star B$
(3) $\bar{A}$ and $\bar{B}$
(4) A^T and B^T
(5) $A^\dagger$ and $B^\dagger$
(6) $(A \star B)^\dagger$

b) Prove that matrix $A = \begin{bmatrix} \cos\vartheta & -\sin\vartheta & 0 \\ \sin\vartheta & \cos\vartheta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is unitary.

c) Find the tensor product

$$\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \otimes \begin{bmatrix} 5 \\ -2 \\ 4 \end{bmatrix}$$

d) Find the tensor product of matrices

$$A \otimes B = \begin{bmatrix} 2+i & 3-i2 \\ 3+i2 & 2 \end{bmatrix} \otimes \begin{bmatrix} i & 3+i2 \\ 3-i2 & 4 \end{bmatrix}$$

e) Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Find $A \otimes (B \otimes C)$