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Deutsch-Jozsa Algorithm

Deutsch algorithm solves a problem on a single input bit in the simple case $f: \{0,1\} \to (0,1)$. The straightforward generalization of this algorithm known as Deutsch-Jozsa algorithm can act on a n-bit function $f: \{0,1\}^n \to \{0,1\}$. This algorithm also determines that f is either constant or f is balanced. The problem here is to determine whether f is constant or balanced.

Consider to solve this problem by a classical algorithm. Suppose we have use the oracle to determine f for exactly half of the possible inputs, you have to made $2^n - 1$ queries to have returned $f(x) = 0$.

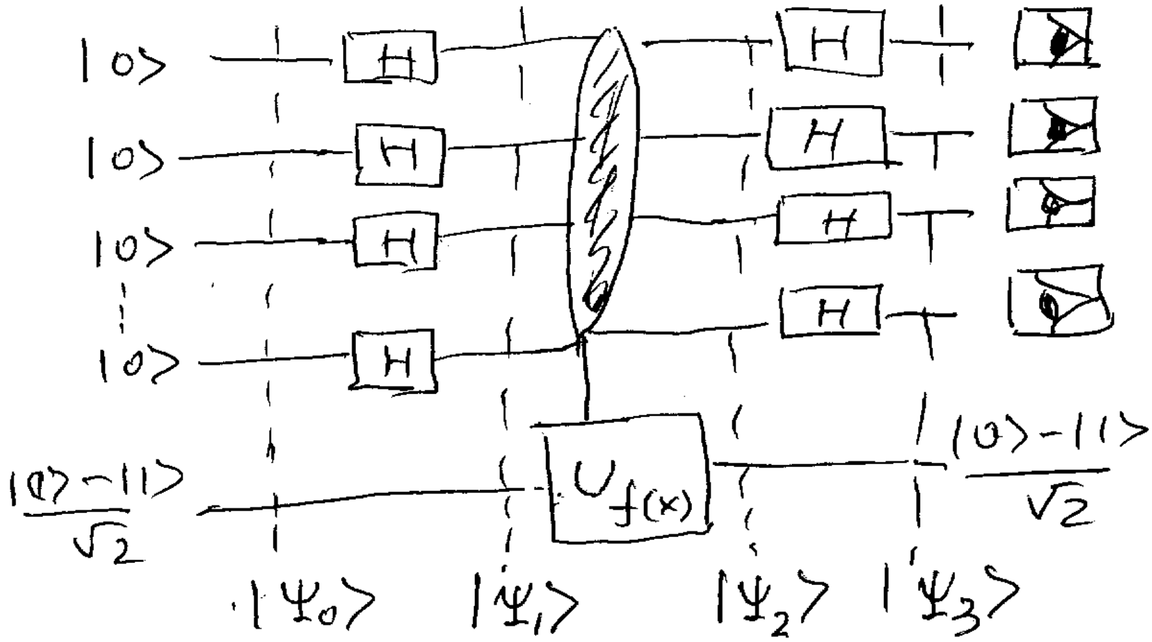
As for the Deutsch algorithm, a quantum algorithm can take advantage of quantum superposition and interference to determine whether f is constant or balanced making only one query to a quantum version of of reversible circuit for f.

Let $\mathbf{x}$ boldface refers to an n-bit string and n qubits in quantum state is $|\mathbf{x}\rangle$. As we did for Deutsch algorithm let define the quantum operator

$$U_f |\mathbf{x}\rangle|y\rangle = |\mathbf{x}\rangle|y \oplus f(x)\rangle$$

and we consider that U_f is one qubit operator $U_{f(x)}$

The quantum circuit for Deutsch-Jozsa algorithm is shown below



① The initial quantum state is

$$|\Psi_0\rangle = \underbrace{|0\rangle|0\rangle \ldots |0\rangle}_{n}\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) = |0\rangle^{\otimes n}\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$$

we use $|0\rangle^{\otimes n}$ to denote the tensor product of n qubits each in the state $|0\rangle$.

② Next step consider the action of n 1-qubit Hadamard gates denoted as $H^{\otimes n}$ on the $|\Psi_0\rangle$ state

$$|\Psi_1\rangle = H^{\otimes n}|\Psi_0\rangle = H^{\otimes n}|0\rangle^{\otimes n}\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) =$$

$$= \left(\frac{1}{\sqrt{2}}\right)^n (|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle \ldots \otimes (|0\rangle + |1\rangle) \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$$

we denote

$$H^{\otimes n}|0\rangle^{\otimes n}\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) = \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$$

Therefore

$$|\Psi_1\rangle = \frac{1}{\sqrt{2}^n} \sum_{x\in\{0,1\}^n} |x\rangle \left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)$$

3 Now consider the state $|\Psi_2\rangle$ after immediately $U_{f(x)}$ gate. This state is

$$|\Psi_2\rangle = U_{f(x)}|\Psi_1\rangle = U_{f(x)}\left(\frac{1}{\sqrt{2}^n} \sum_{x\in\{0,1\}^n} |x\rangle \left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)\right) =$$

$$= \frac{1}{\sqrt{2}^n} \sum_{x\in\{0,1\}^n} (-1)^{f(x)} |x\rangle \left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)$$

4 Now consider the action of $H^{\otimes n}$ Hadamard gates. action of one gate on $|x\rangle$ can be written as

$$H|x\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + (-1)^x|1\rangle\right) =$$

$$= \frac{1}{\sqrt{2}} \sum_{z\in\{0,1\}} (-1)^{xz} |z\rangle$$

Hadamard gates action on an n-qubit basic state is

$$H^{\otimes n}|x\rangle = H^{\otimes n}|x_1\rangle|x_2\rangle\ldots|x_n\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle+(-1)^{x_1}|1\rangle\right)$$

$$\frac{1}{\sqrt{2}}\left(|0\rangle+(-1)^{x_2}|1\rangle\right)\frac{1}{\sqrt{2}}\left(|0\rangle+(-1)^{x_3}|1\rangle\right)\ldots\frac{1}{\sqrt{2}}\left(|0\rangle+(-1)^{x_n}|1\rangle\right) =$$

$$= \frac{1}{\sqrt{2}^n} \sum_{z_1 z_2 \ldots z_n} (-1)^{x_1 z_1 + x_2 z_2 + \ldots + x_n z_n} |z_1\rangle|z_2\rangle\ldots|z_n\rangle$$

$$H^{\otimes n}|x\rangle = \frac{1}{\sqrt{2}^n} \sum_{z\in\{0,1\}^n} (-1)^{x\cdot z} |z\rangle$$

Therefore for $|\Psi_3\rangle$ we have

$$|\Psi_3\rangle = H^{\otimes n}|\Psi_2\rangle = H^{\otimes n}\left[\frac{1}{\sqrt{2^n}}\sum_{x\in\{0,1\}^n}(-1)^{f(x)}|x\rangle\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)\right] =$$

$$= \frac{1}{\sqrt{2^n}}\sum_{x\in\{0,1\}^n}(-1)^{f(x)} H^{\otimes n}|x\rangle\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right) =$$

$$= \frac{1}{\sqrt{2^n}}\sum_{x\in\{0,1\}^n}(-1)^{f(x)}\frac{1}{\sqrt{2^n}}\sum_{z\in\{0,1\}^n}(-1)^{xz}|z\rangle\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right) =$$

$$= \frac{1}{2^n}\sum_{x\in\{0,1\}^n}\left(\sum_{z\in\{0,1\}^n}(-1)^{f(x+xz}|z\rangle\right)\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)$$

If $|z\rangle = |0\rangle^{\otimes n}$

the total amplitud is

$$\frac{1}{2^n}\sum_{x\in\{0,1\}^n}(-1)^{f(x)}$$

consider two case: f constant and f balanced

IF f constatnt the value of amplitude on $|0\rangle^{\otimes n}$ is +1 or −1

therefore it is unity probability of $|0\rangle^{\otimes n}$ on the measurement the function is ~~b~~ constant, otherwise the function is balanced.