Version B

1a) $(x^2-9)y'' - 2y' - 4y = 0$ center $x_0 = 0$

$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0$

$P_0(x) = x^2 - 9$

$x^2 - 9 = 0$

$x = \pm 3$

$\rho = |3-0|$ | $\rho = |-3-0|$

$\rho = 3$ | $\rho = 3$

Interval: $(0-3, 0+3)$

: $\boxed{(-3, 3)}$

1b) $y = \sum_{k=0}^{\infty} C_k x^k$

$y' = \sum_{k=0}^{\infty} k C_k x^{k-1}$

$y'' = \sum_{k=0}^{\infty} k(k-1) C_k x^{k-2}$

$LS = (x^2-9)y'' - 2y' - 4y$

$= (x^2-9)\sum_{k=0}^{\infty} k(k-1)C_k x^{k-2} - 2\sum_{k=0}^{\infty} kC_k x^{k-1} - 4\sum_{k=0}^{\infty} C_k x^k$

$= \underset{Ⓐ}{\sum_{k=0}^{\infty} k(k-1)C_k x^k} - \underset{Ⓑ}{\sum_{k=0}^{\infty} 9k(k-1)C_k x^{k-2}} - \underset{Ⓒ}{\sum_{k=0}^{\infty} 2kC_k x^{k-1}} - \underset{Ⓓ}{\sum_{k=0}^{\infty} 4C_k x^k}$

Ⓑ $\sum_{k=0}^{\infty} 9k(k-1)C_k x^{k-2} \underset{\text{peel}}{=} \underbrace{9(0)(0-1)C_0 x^{0-2}}_{0}{}^{k=0} + \underbrace{9(1)(1-1)C_1 x^{1-2}}_{0}{}^{k=1} + \sum_{k=2}^{\infty} 9k(k-1)C_k x^{k-2}$

$\underset{\text{shift by 2}}{=} \sum_{k=0}^{\infty} 9(k+2)(k+1)C_{k+2} x^k$

Ⓒ $\sum_{k=0}^{\infty} 2kC_k x^{k-1} \underset{\text{peel}}{=} \underbrace{2(0)C_0 x^{0-1}}_{0}{}^{k=0} + \sum_{k=1}^{\infty} 2kC_k x^{k-1} \underset{\text{shift by 1}}{=} \sum_{k=0}^{\infty} 2(k+1)C_{k+1} x^k$

Ⓐ - Ⓑ - Ⓒ - Ⓓ

$LS = \sum_{k=0}^{\infty} [\,k(k-1)C_k x^k - \sum_{k=0}^{\infty} 9(k+2)(k+1)C_{k+2} x^k - \sum_{k=0}^{\infty} 2(k+1)C_{k+1} x^k - \sum_{k=0}^{\infty} 4C_k x^k$

$$\boxed{LS = \sum_{k=0}^{\infty} \left[-9(k+2)(k+1)C_{k+2} - 2(k+1)C_{k+1} + (k(k-1)-4)C_k\right] x^k = 0}$$

1c) $-9(k+2)(k+1)C_{k+2} - 2(k+1)C_{k+1} + (k(k-1)-4)C_k = 0$

$$\boxed{C_{k+2} = \frac{2(k+1)C_{k+1} - (k(k-1)-4)C_k}{-9(k+2)(k+1)}}$$

1d) $C_0 = C_0$
$C_1 = C_1$

$k=0 \rightarrow C_{0+2} = \dfrac{2(0+1)C_{0+1} - (0(0-1)-4)C_0}{-9(0+2)(0+1)}$

$C_2 = \dfrac{2C_1 + 4C_0}{-18}$

$C_2 = -\frac{1}{9}C_1 - \frac{2}{9}C_0$

$k=1 \rightarrow C_{1+2} = \dfrac{2(1+1)C_{1+1} - (1(1-1)-4)C_1}{-9(1+2)(1+1)}$

$C_3 = \dfrac{4C_2 + 4C_1}{-54}$

$C_3 = -\frac{2}{27}C_2 - \frac{2}{27}C_1$

$C_3 = -\frac{2}{27}\left(-\frac{1}{9}C_1 - \frac{2}{9}C_0\right) - \frac{2}{27}C_1$

$C_3 = \frac{-16}{243}C_1 + \frac{4}{243}C_0$

$k=2 \rightarrow C_{2+2} = \dfrac{2(2+1)C_{2+1} - (2(2-1)-4)C_2}{-9(2+2)(2+1)}$

$C_4 = \dfrac{6C_3 + 2C_2}{-108}$

$C_4 = -\frac{1}{18}C_3 - \frac{1}{54}C_2$

$C_4 = -\frac{1}{18}\left(\frac{-16}{243}C_1 + \frac{4}{243}C_0\right) - \frac{1}{54}\left(-\frac{1}{9}C_1 - \frac{2}{9}C_0\right)$

$C_4 = \frac{25}{4374}C_1 + \frac{7}{2187}C_0$

$$y = \sum_{k=0}^{\infty} C_k x^k = C_0 + C_1x + C_2x^2 + C_3x^3 + C_4x^4 + \dots$$

$$= C_0 + C_1x + \left(-\frac{1}{9}C_1 - \frac{2}{9}C_0\right)x^2 + \left(-\frac{16}{243}C_1 + \frac{4}{243}C_0\right)x^3 + \left(\frac{25}{4374}C_1 + \frac{7}{2187}C_0\right)x^4 + \dots$$

$$= \boxed{C_0\left(1 - \frac{2}{9}x^2 + \frac{4}{243}x^3 + \frac{7}{2187}x^4 + \dots\right) + C_1\left(x - \frac{1}{9}x^2 - \frac{16}{243}x^3 + \frac{25}{4374}x^4 + \dots\right)}$$

1e) $y = C_0\left(1 - \frac{2}{9}x^2 + \frac{4}{243}x^3 + \frac{7}{2187}x^4 + \dots\right) + C_1\left(x - \frac{1}{9}x^2 - \frac{16}{243}x^3 + \frac{25}{4374}x^4 + \dots\right)$

$y(0) = -1$

$y = C_0\left(1 - \frac{2}{9}(0)^2 + \frac{4}{243}(0)^3 + \frac{7}{2187}(0)^4 + \dots\right) + C_1\left(0 - \frac{1}{9}(0)^2 - \frac{16}{243}(0)^3 + \frac{25}{4374}(0)^4 + \dots\right)$

$C_0 = -1$

$y' = C_0\left(-\frac{4}{9}x + \frac{4}{81}x^2 + \frac{218}{2187}x^3 + \dots\right) + C_1\left(1 - \frac{2}{9}x - \frac{16}{81}x^2 + \frac{50}{2187}x^3 + \dots\right)$

$y'(0) = 2$

$y' = C_0\left(-\frac{4}{9}(0) + \frac{4}{81}(0)^2 + \frac{218}{2187}(0)^3 + \dots\right) + C_1\left(1 - \frac{2}{9}(0) - \frac{16}{81}(0)^2 + \frac{50}{2187}(0)^3 + \dots\right)$

$C_1 = 2$

$y = -\left(1 - \frac{2}{9}x^2 + \frac{4}{243}x^3 + \frac{7}{2187}x^4 + \dots\right) + 2\left(x - \frac{1}{9}x^2 - \frac{16}{243}x^3 + \frac{25}{4374}x^4 + \dots\right)$

$\boxed{y = -1 + 2x - \frac{4x^3}{27} + \frac{2x^4}{243}}$