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Test #3 Formula Sheet
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☆ $(2+x)y'' - (1+x)y' + 3y = 0$

In this problem you will find a series solution

(a) Analyze singular points to determined an interval on which the series solution must converge.

$P_0(x) = 2+x$ $\quad \rho = $ distance from 0 to $-2 = 2$

$2+x=0$

$x=-2 \Rightarrow$ Interval : $(0-2, 0+2)$

$= (-2, 2)$ Interval

In $(-2,2)$ the series solution must converge.

B) Let $y = \sum_{k=0}^{\infty} c_k x^k \Rightarrow y' = \sum_{k=0}^{\infty} k c_k x^{k-1} \Rightarrow y'' = \sum_{k=0}^{\infty} k(k-1)c_k x^{k-2}$

Plug in to ☆

$(2+x)\sum_{k=0}^{\infty} k(k-1)c_k x^{k-2} \underbrace{-(1+x)}_{(-1-x)}\sum_{k=0}^{\infty} k c_k x^{k-1} + \sum_{k=0}^{\infty} 3c_k x^k$ (C)

$\underbrace{\sum_{n=0}^{\infty} 2k(k-1)c_k x^{k-2}}_{A} + \underbrace{\sum_{k=0}^{\infty} k(k-1)c_k x^{k-1}}_{B} - \underbrace{\sum_{k=0}^{\infty} k c_k x^{k-1}}_{C} - \underbrace{\sum_{k=0}^{\infty} k c_k x^k}_{D} + \underbrace{\sum_{k=0}^{\infty} 3c_k x^k}_{E}$

c) ☆ All must have x^k $\quad$ D - OK - E

A) $\sum_{k=0}^{\infty} 2k(k-1)c_k x^{k-2}$ (peel by 2) $= \underset{k=0}{0} + \underset{k=1}{0} + \sum_{k=2}^{\infty} 2k(k-1)c_k x^{k-2} = \sum_{k=0}^{\infty} 2(k+2)(k+1)c_{k+2}x^k$ (shift by 2)

B) $\sum_{k=0}^{\infty} k(k-1)c_k x^{k-1}$ (peel by 1) $= \underset{k=0}{0} + \sum_{k=1}^{\infty} k(k-1)c_k x^{k-1} = \sum_{k=0}^{\infty} k(k+1)c_{k+1}x^k$ (shift by 1)

C) $\sum_{k=0}^{\infty} k c_k x^{k-1}$ (peel by 1) $= \underset{k=0}{0} + \sum_{k=1}^{\infty} k c_k x^{k-1} = \sum_{k=0}^{\infty} (k+1)c_{k+1}x^k$ (shifting by one)

Plug them back

$\sum_{k=0}^{\infty} 2(k+2)(k+1)c_{k+2}x^k + \sum_{k=0}^{\infty} k(k+1)c_{k+1}x^k - \sum_{k=0}^{\infty}(k+1)c_{k+1}x^k - \sum_{k=0}^{\infty} k c_k x^k + \sum_{k=0}^{\infty} 3c_k x^k$

$\sum_{k=0}^{\infty}\left[2(k+2)(k+1)c_{k+2} + k(k+1)c_{k+1} - (k+1)c_{k+1} - kc_k + 3c_k\right]x^k$

$k=0$

$2(0+2)(0+1)c_{0+2} + 0(0+1)c_{0+1} - (0+1)c_{0+1} - 0c_0 + 3c_0$

$2(2)(1)c_2 + 0$

$4c_2 - c_1 + 3c_0 = 0$ $\quad c_2 = \frac{1}{4}c_1 - \frac{3}{4}c_0$

$4c_2 = \frac{c_1 - 3c_0}{4}$

$\sum_{k=1}^{\infty}\left[2(k+2)(k+1)c_{k+2} + k(k+1)c_{k+1} - (k+1)c_{k+1} - kc_k + 3c_k\right]x^k$

change $k=n$

$\sum_{n=1}\left[2(n+2)(n+1)c_{n+2} + n(n+1)c_{n+1} - (n+1)c_{n+1} - nc_n + 3c_n\right]x^k$

Simply $\quad$ $-c_n(n-3)$ Rerange

$2(n+2)(n+1)c_{n+2} + (n-1)(n+1)c_{n+1} - (n-3)c_n = 0$

To find c_{n+2}

$2(n+2)(n+1)c_{n+2} = \frac{-(n-1)(n+1)c_{n+1} + (n-3)c_n}{2(n+2)(n+1)}$

$c_{n+2} = \frac{-(n-1)(n+1)c_{n+1} + (n-3)c_n}{2(n+2)(n+1)}$

D) $n=0$

$c_{0+2} = \frac{-(0-1)(0+1)c_{0+1} + (0-3)c_0}{2(0+2)(0+1)} = c_2 = \frac{c_1 - 3c_0}{4}$

$c_2 = \frac{1}{4}c_1 - \frac{3}{4}c_0$

$n=1$

$c_{1+2} = \frac{-(1-1)(1+1)c_{1+1} + (1-3)c_1}{2(1+2)(1+1)} = \frac{0c_2 - 2c_1}{12} = c_3 = -\frac{1}{6}c_1$

$n=2$

$c_{2+2} = \frac{-(2-1)(2+1)c_{2+1} + (2-3)c_2}{2(2+2)(2+1)} = \frac{-3c_3 - c_2}{24} = -\frac{1}{8}c_3 - \frac{1}{24}c_2$

$-\frac{1}{8}\left(-\frac{1}{6}c_1\right) - \frac{1}{24}\left(\frac{1}{4}c_1 - \frac{3}{4}c_0\right) = \frac{1}{48}c_1 - \frac{1}{96}c_1 + \frac{1}{32}c_0$

$= \frac{1}{96}c_1 + \frac{1}{32}c_0 = c_4$

$y = \sum_{k=0}^{\infty} c_k x^k = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4$ ☆!!

$= c_0 + c_1x + \left(\frac{1}{4}c_1 - \frac{3}{4}c_0\right)x^2 + \left(-\frac{1}{6}c_1\right)x^3 + \left(\frac{1}{96}c_1 + \frac{1}{32}c_0\right)x^4$

e) $y(0)=4$, $y'(0)=3$

Since $c_0=4$, then $c_1=3$

Plug c_1 & c_2 to ☆!!

$y = 4 + 3x + \left(\frac{1}{4}(3) - \frac{3}{4}(4)\right)x^2 + \left(-\frac{1}{6}(3)\right)x^3 + \left(\frac{1}{96}(3) + \frac{1}{32}(4)\right)x^4$

$$\boxed{y = 4 + 3x - \frac{9}{4}x^2 - \frac{1}{2}x^3 + \frac{1}{24}x^4}$$