Navier-Stokes Equations: A Confluence of Fluid Dynamics and Mathematical Complexity

The Navier-Stokes equations, named after Claude-Louis Navier and George Gabriel Stokes, are among the most fundamental tools used to understand fluid dynamics and simulate fluid flow. It was first introduced in 1822 by Navier and continued by Stokes in 1842. These mathematical equations describe how the velocity, pressure, temperature, and density of a moving fluid are interrelated, applying both to everyday fluids such as water and air, as well as more complex fluids like plasma (Batchelor, G.K. 2000). The Navier-Stokes equation can also be applied to solids in some cases, an example of this glacier flowing down the mountain (Numberphile, 2019). In a sense the term 'liquid' could be used to describe something that changes to match the container that contains it (Numberphile, 2019).

These equations are nonlinear partial differential equations that consist of conservation of momentum and conservation of mass principles, derived from Newton's second law of motion. By making use of the assumptions of continuum mechanics, which holds that matter is continuously distributed and mathematical properties can be averaged over a small volume, the equations provide a mathematical model for the motion of fluids (Pope, S.B., 2000).

The Navier-Stokes equations can be written in a vector form as:

$$\nabla u = 0$$

$$\rho \, (Du/Dt) = -\nabla p + \mu \nabla^2 u + \rho F$$

where:

ρ: Fluid density

u: Fluid velocity vector

p: Pressure

μ: Dynamic viscosity

g: External force acting on the liquid

The first equation ($\nabla u = 0$), sometimes called the incompressibility represents conservation of mass. This means that even if the fluid changes velocity or shape, the mass will remain constant. The 'u' in the second equation is vector or speed with direction. It could be rewritten as u=(u,v,w) which represents the component in the x direction, y direction, and the z direction. The symbol '∇' or nabla is a differential operator to the three components of the vector, meaning the three components must be differentiated. The way to differentiate the three components is as follows:

$$\nabla u = (du/dx) + (dv/dy) + (dw/dz) = 0$$

The extended equation shows how each of the components (u, v, and w) changes in their respective directions (x, y, and z).

The second equation ($\rho\ (Du/Dt) = -\nabla p + \mu\nabla^2 u + \rho F$) is an extension of Newton's second law which states the acceleration of an object is directly proportional to the acting force and inversely proportional to its mass. The equation is in a way similar to the famous F = ma formula. ρ or the fluid density consist of mass over volume, is acting as the mass in the Navier-Stokes formula. Du/Dt is acting as the acceleration in this case. This is because a time derivative of velocity (which in the case Navier-Stokes equation is 'u') is equal to acceleration.

The right-hand side of the equation which is $-\nabla p + \mu\nabla^2 u + \rho F$ represents all the forces present on the liquid. It is separated into two kinds of force, $-\nabla p + \mu\nabla^2 u$ is the internal forces acting on the liquid and ρF the external force of the liquid. ∇p is the gradient of pressure, representing the change in pressure, it causes the air to move from high pressure to low pressure and generating force. $\mu \nabla^2 u$ represents viscosity, as the fluid moves or changes shape countless numbers of molecules colliding and sliding with each other generating friction. Fluid like air that moves around freely has low viscosity while something like glue or honey which is sticky and thick has a higher viscosity. The final character which is ρF is simply the external force acting on the liquid, in the majority of cases it is changed to gravity.

Despite their concise form, the Navier-Stokes equations are immensely complex. While in theory could be applied in almost all situations involving a fluid, it does not always provide a solution when applied. In a normal equation when there's input there will be a way to solve it and a solution will come out as an output. However, the Navier-Stokes equation is so complex that it could not be consistently applied to all situations which creates many issues and problems. One such problem pertains to the existence and smoothness of solutions in three dimensions over time, which remains unresolved despite being posed over a century ago. The Clay Mathematics Institute even designated it as one of the seven "Millennium Prize Problems," offering a million-dollar reward for a solution (Fefferman, C., 2006).

The complexity of these equations stems from their nonlinearity, which makes exact solutions limited to specific conditions. Moreover, nonlinearity brings about phenomena like turbulence, which is ubiquitous in nature but not yet fully understood. This lack of understanding leads to substantial approximation in engineering applications, such as weather prediction, aeronautics, and oceanography, where solving the equations becomes a matter of computational fluid dynamics (Lesieur, M., 2008,). One way to ensure that Navier-Stokes equation produces a solution is to make simplifications or assumptions such as removing time from the equation or making assumptions to reduce some of the terms (Numberphile, 2019). Another method to ensure solution is called Reynolds averaging, the method is to calculate the average liquid velocity instead of having velocity field defined everywhere on the fluid. Reynolds averaging is usually used in climate modeling because it is simply not possible to simulate every particle in the atmosphere.

Despite the difficulties associated with their resolution, the Navier-Stokes equations have proven instrumental in numerous fields. For instance, their solutions are crucial for aerodynamic simulations in the design of aircraft, for understanding weather patterns, modeling ocean currents, and studying blood flow in biological systems (Tritton, D.J., 1988).

In conclusion, the Navier-Stokes equations are a cornerstone of fluid dynamics. They encapsulate the profound intricacy of fluid motion, demonstrating how this fundamental physical phenomenon intertwines with complex mathematical structures. These equations, despite their unresolved theoretical problems and computational challenges, remain a critical tool for scientists and engineers striving to understand and harness the power of fluid motion.

Sources:

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