Ken Mei

Test #3 Solutions

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In this problem you will find a series solution of the differential equation $(x^2-4)y'' - 3y' - y = 0$ centered at 0.

Part (a)

$$P_0(x) = x^2 - 4$$

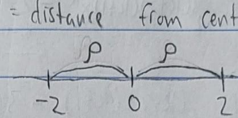
$$x^2 - 4 = 0$$

$$+4 \quad +4$$

$$\sqrt{x^2} = \pm\sqrt{4}$$

$$x = \pm 2$$

ρ = distance from center = 2



$$(0-2,\ 0+2) = (-2,2)$$

$$\boxed{\text{Interval is } (-2,2)}$$

Part (b)

Let $y = \sum_{k=0}^{\infty} c_k x^k$

$$y = \sum_{k=0}^{\infty} c_k x^k \quad \Big| \quad y' = \sum_{k=0}^{\infty} c_k \cdot k \cdot x^{k-1} \quad \Big| \quad y'' = \sum_{k=0}^{\infty} c_k \cdot k \cdot (k-1) \cdot x^{k-2}$$

$$(x^2-4) \cdot \sum_{k=0}^{\infty} c_k \cdot k \cdot (k-1) \cdot x^{k-2} - 3 \cdot \sum_{k=0}^{\infty} c_k \cdot k \cdot x^{k-1} - \sum_{k=0}^{\infty} c_k x^k$$

$$\underbrace{\sum_{k=0}^{\infty} c_k \cdot k(k-1)\, x^k}_{A} - \underbrace{\sum_{k=0}^{\infty} 4c_k k(k-1) x^{k-2}}_{B} - \underbrace{\sum_{k=0}^{\infty} 3c_k k x^{k-1}}_{C} - \underbrace{\sum_{k=0}^{\infty} c_k x^k}_{D}$$

B $\displaystyle\sum_{k=0}^{\infty} 4c_k k(k-1) x^{k-2} = \overset{k=0}{0} + \overset{k=1}{0} + \sum_{k=2}^{\infty} 4c_k k(k-1)\, x^{k-2}$

peel of by 2

$$= \sum_{k=0}^{\infty} 4c_{k+2} (k+2)(k+1)\, x^k$$

Shifting B by 2

Ken mei Test #3 Solutions version C

Part (b)

ⓒ $\sum_{k=0}^{\infty} 3c_k k x^{k-1} = \underset{\text{peel of by 1}}{\overset{k=0}{0}} + \sum_{k=1}^{\infty} 3c_k k x^{k-1} = \sum_{k=0}^{\infty} 3c_{k+1}(k+1)x^k$

now shifting ⓒ by 1

$$\sum_{k=0}^{\infty} c_k k(k-1)x^k - \sum_{k=0}^{\infty} 4c_{k+2}(k+2)(k+1)x^k - \sum_{k=0}^{\infty} 3c_{k+1}(k+1)x^k - \sum_{k=0}^{\infty} c_k x^k$$

$$\sum_{k=0}^{\infty}\left[c_k k(k-1) - 4c_{k+2}(k+2)(k+1) - 3c_{k+1}(k+1) - c_k\right]x^k = 0$$

$$k=0,\ \underbrace{c_0(0)(0-1)}_{0} - \underbrace{4c_{0+2}(0+2)(0+1)}_{-8c_2} - \underbrace{3c_{0+1}(0+1)}_{-3c_1} - \underbrace{c_0}_{-1c_0}$$

$-1c_0 - 3c_1 - 8c_2 \rightarrow$ ①

$$\sum_{k=0}^{\infty}\left[c_k k(k-1) - \overbrace{4c_{k+2}(k+2)(k+1)}^{④} - \overbrace{3c_{k+1}(k+1)}^{③} - c_k\right]x^k = 0$$

$c_k k(k-1) - c_k = [k^2 - k - 1]c_k \rightarrow$ ②

$$① + \sum_{k=1}^{\infty}\left[② + ③ + ④\right]x^k = 0$$

$$-1c_0 - 3c_1 - 8c_2 + \sum_{k=1}^{\infty}\left[(k^2-k-1)c_k - 3c_{k+1}(k+1) - 4c_{k+2}(k+2)(k+1)\right]x^k = 0$$

Ken Mei Test #3 Solutions Version C

Part (c) Going back to part (b) $\rightarrow$ 1

$$1 \rightarrow -\cancel{1}c_0 - \cancel{3}c_1 - 8c_2 = 0$$

$$+\cancel{1}c_0 + \cancel{3}c_1 \qquad +1c_0 + 3c_1$$

$$\frac{\cancel{-8}c_2}{\cancel{-8}} = \frac{1c_0 + 3c_1}{-8}$$

$$\boxed{c_2 = -\frac{1}{8}c_0 - \frac{3}{8}c_1}$$

Going back to part (b) $\rightarrow \sum_{k=1}^{\infty}\left[2 + 3 + 4\right]x^k = 0$

$$\cancel{(k^2-k-1)c_k} \cancel{- 3c_{k+1}(k+1)} - 4c_{k+2}(k+2)(k+1) = 0$$

$$\cancel{-(k^2-k-1)c_k} \cancel{+3c_{k+1}(k+1)} \qquad -(k^2-k-1)c_k + 3c_{k+1}(k+1)$$

$$\frac{\cancel{-4}c_{k+2}\cancel{(k+2)(k+1)}}{\cancel{-4(k+2)(k+1)}} = \frac{-(k^2-k-1)c_k + 3c_{k+1}(k+1)}{-4(k+2)(k+1)}$$

$$\boxed{c_{k+2} = \frac{-(k^2-k-1)c_k + 3c_{k+1}(k+1)}{-4(k+2)(k+1)} \quad \text{for } k \geq 1}$$

Part (d) $c_0 = c_0,\ c_1 = c_1$

$$k=0,\quad c_{0+2} = \frac{-(0^2-0-1)c_0 + 3c_{0+1}(0+1)}{-4(0+2)(0+1)} = c_2 = \frac{c_0 + 3c_1}{-8} \rightarrow \boxed{c_2 = -\frac{1}{8}c_0 - \frac{3}{8}c_1}$$

$$k=1,\quad c_{1+2} = \frac{-(1^2-1-1)c_1 + 3c_{1+1}(1+1)}{-4(1+2)(1+1)} = c_3 = \frac{-1}{24}c_1 - \frac{6}{24}c_2$$

Solving for c_3 $\rightarrow$ $\frac{-1}{24}c_1 - \frac{6}{24}\left(\frac{-1}{8}c_0 - \frac{3}{8}c_1\right) \rightarrow -\frac{1}{24}c_1 + \frac{6}{192}c_0 + \frac{18}{192}c_1$

$$\left(\frac{8}{8}\right)\cdot\frac{-1}{24}c_1 + \frac{18}{192}c_1 = \frac{-8}{192}c_1 + \frac{18}{192}c_1 = \frac{10}{192}c_1$$

$$\boxed{c_3 = \frac{6}{192}c_0 + \frac{10}{192}c_1}$$

Ken mei Test # 3 Solutions Version C

$$k=2,\ c_{2+2} = \frac{-(2^2-2-1)c_2 + 3c_{2+1}(2+1)}{-4(2+2)(2+1)} = c_4 = \frac{1}{48}c_2 - \frac{9}{48}c_3$$

$$\frac{1}{48}\left(\frac{-1}{8}c_0 - \frac{3}{8}c_1\right) - \frac{9}{48}\left(\frac{6}{192}c_0 + \frac{10}{192}c_1\right)$$

$$\frac{-1}{384}c_0 - \frac{3}{384}c_1 \quad - \frac{54}{9216}c_0 - \frac{90}{9216}c_1$$

$$\left(\frac{24}{24}\right)\cdot\frac{-1}{384}c_0 - \frac{54}{9216}c_0 \quad \left(\frac{24}{24}\right)\cdot\frac{-3}{384}c_1 - \frac{90}{9216}c_1$$

$$\frac{-24}{9216}c_0 - \frac{54}{9216}c_0 \quad \frac{-72}{9216}c_1 - \frac{90}{9216}c_1$$

$$\boxed{c_4 = \frac{-78}{9216}c_0 - \frac{162}{9216}c_1}$$

Part (e) Consider the above differential equation together with initial values:
$y(0) = 1,\ y'(0) = 2$

Lets say $y = \sum_{k=0}^{\infty} c_k x^k = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots$

when $y(0) = c_0$ and $y(0) = 1$ so $\boxed{c_0 = 1}$

$y' = \sum_{k=0}^{\infty} c_k k x^{k-1} = c_1 + 2c_2 x + 3c_3 x^2 + \ldots$

when $y'(0) = c_1$ and $y'(0) = 2$ so $\boxed{c_1 = 2}$

Ken mei Test # 3 Solutions Version C

Part (e)

$c_0 = \boxed{1}$

$c_1 = \boxed{2}\ x$

$c_2 = -\frac{1}{8}c_0 - \frac{3}{8}c_1 \rightarrow -\frac{1}{8}(1) - \frac{3}{8}(2) = -\frac{1}{8} - \frac{6}{8} = \boxed{\frac{-7}{8}}\ x^2$

$c_3 = \frac{6}{192}c_0 + \frac{10}{192}c_1 \rightarrow \frac{6}{192}(1) + \frac{10}{192}(2) = \frac{6}{192} + \frac{20}{192} = \boxed{\frac{26}{192}}\ x^3$

$c_4 = \frac{-78}{9216}c_0 - \frac{162}{9216}c_1 \rightarrow \frac{-78}{9216}(1) - \frac{162}{9216}(2) = \frac{-78}{9216} - \frac{324}{9216}$

$\frac{-78}{9216} - \frac{324}{9216} = \boxed{\frac{-402}{9216}}\ x^4$

General Solution: $y = 1 + 2x - \frac{7}{8}x^2 + \frac{26}{192}x^3 - \frac{402}{9216}x^4 + \ldots$