Ken Mei

Ken Mei Test # 3 cheat Sheet | In this problem you will solve the DE ☆ $\rightarrow (x+6)y'' - (9-x)y' + y = 0$?

(1) Interval the DE will converge

$P_0(x) = x+6$ ρ = distance from 0 to $-6 = 6$

$x+6=0 \rightarrow x=-6$, Interval: $(0-6, 0+6)$

$= (-6,6) \rightarrow$ Interval

(2) substituting $y = \sum_{k=0}^{\infty} c_k x^k$ into ☆, you get that; $y = \sum_{k=0}^{\infty} c_k x^k = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots$

$y' = \sum_{k=0}^{\infty} c_k \cdot k\, x^{k-1}$, $y'' = \sum_{k=0}^{\infty} c_k \cdot k \cdot (k-1)\, x^{k-2}$, Now plug y'', y', and y back into ☆ to get

$(x+6)\sum_{k=0}^{\infty} c_k k(k-1)x^{k-2} - (9-x)\sum_{k=0}^{\infty} c_k \cdot k\, x^{k-1} + \sum_{k=0}^{\infty} c_k x^k$, Then distribute and you will have 5 terms

distribution: $\underbrace{\sum_{k=0}^{\infty} c_k k(k-1)x^{k-1}}_{A} + \underbrace{\sum_{k=0}^{\infty} 6c_k k(k-1)x^{k-2}}_{B} - \underbrace{\sum_{k=0}^{\infty} 9c_k k\, x^{k-1}}_{C} + \underbrace{\sum_{k=0}^{\infty} c_k k x^k}_{D} + \underbrace{\sum_{k=0}^{\infty} c_k x^k}_{E}$, Now we want A, B, C to have x^k

(A) $\sum_{k=0}^{\infty} c_k k(k-1)x^{k-1} \underset{\text{peel}}{=} \overset{k=0}{0} + \sum_{k=1}^{\infty} c_k k(k-1)x^{k-1} = \underset{\text{shift by 1}}{\sum_{k=0}^{\infty}} c_{k+1}(k+1)k\, x^k$

(B) $\sum_{k=0}^{\infty} 6c_k k(k-1)x^{k-2} \underset{\text{peel off by 2}}{=} \overset{k=0}{0} + \overset{k=1}{0} + \sum_{k=2}^{\infty} 6c_k k(k-1)x^{k-2} \underset{\text{shift by 2}}{=} \sum_{k=0}^{\infty} 6c_{k+2}(k+2)(k+1)x^k$

(C) $\sum_{k=0}^{\infty} 9c_k k\, x^{k-1} \underset{\text{peel}}{=} \overset{k=0}{0} + \sum_{k=1}^{\infty} 9c_k k\, x^{k-1} = \underset{\text{shift by 1}}{\sum_{k=0}^{\infty}} 9c_{k+1}(k+1)x^k$

$\sum_{k=0}^{\infty} c_{k+1}(k+1)k x^k + \sum_{k=0}^{\infty} 6c_{k+2}(k+2)(k+1)x^k - \sum_{k=0}^{\infty} 9c_{k+1}(k+1)x^k + \sum_{k=0}^{\infty} c_k k x^k + \sum_{k=0}^{\infty} c_k x^k$

factored out x^k.

$\sum_{k=0}^{\infty}\left[c_{k+1}(k+1)k + 6c_{k+2}(k+2)(k+1) - 9c_{k+1}(k+1) + c_k k + c_k\right]x^k = 0 \rightarrow$ ☆☆

Still Part (2), $k=0 \rightarrow$ ☆☆ to get

$c_{0+1}(0+1)(0) + 6c_{0+2}(0+2)(0+1) - 9c_{0+1}(0+1) + c_0(0) + c_0$

$= c_1(1)(0) + 6c_2(2)(1) - 9c_1(1) + c_0(0) + c_0$

$= 1c_0 - 9c_1 + 12c_2 \rightarrow ①$, now combine like terms on ☆☆

$\sum_{k=0}\left[c_{k+1}(k+1)k + 6c_{k+2}(k+2)(k+1) - 9c_{k+1}(k+1) + c_k k + c_k\right]$

$(c_k k + c_k) = (k+1)c_k \rightarrow ①$, $[k(k+1) - 9(k+1)]c_{k+1}$

$6(k+2)(k+1)c_{k+2} \rightarrow ③$, $= (k^2 - 8k - 9)c_{k+1} \rightarrow ②$

To finish part (2) combine ① $+ \sum_{k=1}^{\infty}[① + ② + ③]x^k = 0$

(3), (a) Find the recurrence relation? Going back to ① set $= 0$

$1c_0 - 9c_1 + 12c_2 = 0 \rightarrow \frac{12c_2}{12} = \frac{-1c_0 + 9c_1}{12} \rightarrow \boxed{c_2 = \frac{-1c_0 + 9c_1}{12}}$

$-1c_0 + 9c_1$

(3), (b) $\rightarrow$ Going back to $\sum_{k=1}^{\infty}[① + ② + ③]x^k = 0$

$\cancel{(k+1)c_k} + \cancel{(k^2-8k-9)c_{k+1}} + 6(k+2)(k+1)c_{k+2} = 0$

$\cancel{-(k+1)c_k} - \cancel{(k^2-8k-9)c_{k+1}} \rightarrow$ subtract on both sides

$-(k^2-8k-9)c_{k+1} - (k+1)c_k \rightarrow$ expand $-(k^2-8k-9)c_{k+1}$ to get $(k-9)(k+1)$ which will help us cancel out $(k+1)$ later

$-(k^2+k-9k-9)c_{k+1}$

$-(k^2+k) + (-9k-9)c_{k+1}$

$-1[k(k+1) - 9(k+1)]c_{k+1} - (k+1)c_k$

$c_{k+2} = \frac{-1[(k-9)\cancel{(k+1)}]c_{k+1} - \cancel{(k+1)}c_k}{6(k+2)\cancel{(k+1)}}$

$\boxed{c_{k+2} = \frac{-(k-9)c_{k+1} - c_k}{6(k+2)}}$ for $\boxed{n \geq 1}$

(4) The general solution to $(x-6)y'' - (9-x)y' + y = 0$ converges at least on $(-6,6)$ and is

$n=0$ $\boxed{c_0 = c_0}$, $\boxed{c_1 = c_1}$

$c_{k+2} = \frac{-(k-9)c_{k+1} - c_k}{6(k+2)}$, $c_{0+2} = \frac{-(0-9)c_{0+1} - c_0}{6(0+2)} = c_2$

$\boxed{c_2 = \left(\frac{9c_1}{12} - \frac{1c_0}{12}\right)x^2}$, $n=1$ $c_{1+2} = \frac{-(1-9)c_{1+1} - c_1}{6(1+2)} = c_3$

$c_3 = \frac{8c_2 - c_1}{18} \rightarrow c_3 = \frac{1}{18}\left(8\left(\frac{9c_1}{12} - \frac{c_0}{12}\right) - c_1\right) \rightarrow \frac{8}{18}\left(\frac{9c_1}{12} - \frac{c_0}{12}\right) - \frac{c_1}{18}$

$\frac{72}{216}c_1 - \frac{8c_0}{216} - \frac{c_1}{18}$, Note: $\frac{72}{216} = \frac{1}{3}$ and $\frac{-8}{216} = \frac{-1}{27}$, so $\frac{1}{3}c_1 - \frac{c_1}{18} = \frac{18}{54}c_1 - \frac{3c_1}{54} = \frac{15c_1}{54}$ this means $\boxed{c_3 = \left(\frac{15}{54}c_1 - \frac{1}{27}c_0\right)x^3}$

$n=2 \rightarrow c_{2+2} = \frac{-(2-9)c_{2+1} - c_2}{6(2+2)} = c_4$, $c_4 = \frac{7c_3 - c_2}{24}$ or $c_4 = \frac{7c_3}{24} - \frac{c_2}{24}$, substitute the values I found for c_3 and c_2 in for c_4 to solve

$c_4 = \frac{7\left(\frac{15}{54}c_1 - \frac{1}{27}c_0\right)}{24} - \frac{\left(\frac{9}{12}c_1 - \frac{1c_0}{12}\right)}{24}$, Note: $\frac{105}{54} = \frac{35}{18}$

$\frac{\frac{105}{54}c_1 - \frac{7}{27}c_0}{24} = \frac{1}{24}\left(\frac{35}{18}c_1 - \frac{7}{27}\right) - \frac{1}{24}\left(\frac{9}{12}c_1 - \frac{c_0}{12}\right) =$

$= \frac{35c_1}{432} - \frac{7c_0}{648} - \frac{9}{288}c_1 + \frac{1}{288}c_0$, Note: $\frac{-9}{288} = -\frac{1}{32}$

$\frac{35c_1}{432} - \frac{1}{32}c_1 = \frac{43}{864}c_1$, $\frac{-7}{648}c_0 + \frac{1}{288}c_0 = \frac{-19c_0}{2592}$

$\boxed{c_4 = \left(\frac{43}{864}c_1 - \frac{19}{2592}c_0\right)x^4}$

General Solution for Part (4) $y = c_0\left(1 - \frac{1}{12}x^2 - \frac{1}{27}x^3 - \frac{19}{2592}x^4 + \dots\right) + c_1\left(1x + \frac{9}{12}x^2 + \frac{15}{54}x^3 + \frac{43}{864}x^4 + \dots\right)$