Given that the differential equation is

$$(x+3)y'' - (11-x)y' + y = 0 \quad \cdots \quad (i)$$

(1). For singular points of equn (i), we have $x+3=0$.
$\Rightarrow x=-3$.

Therefore the minimum value of the radius of convergence of the power series solution of equn (i) about the point $x_0=0$ is given by

$$R = \min\{|x_0 - x_1| : x_1 \text{ is singular point}\}$$
$$= \min\{|0+3|\}$$
$$= \min\{3\}$$
$$= 3.$$

So the interval of convergence is $(-R, R) = (-3, 3)$.

$\therefore$ By analyzing the singular points of the differential equation, we know that a series solution of the form $y = \sum_{K=0}^{\infty} C_K x^K$ for the differential equn will converge atleast on the interval $\boxed{(-3, 3)}$

(2). Substitute $y = \sum_{K=0}^{\infty} C_K x^K$ into the given differential equn.

$\therefore y' = \sum_{K=1}^{\infty} K C_K x^{K-1}$ and $y'' = \sum_{K=2}^{\infty} K(K-1) C_K x^{K-2}$.

Therefore we have-

$$(x+3)\sum_{k=2}^{\infty} k(k-1)c_k x^{k-2} - (11-x)\sum_{k=1}^{\infty} k\cdot c_k x^{k-1} + \sum_{k=0}^{\infty} c_k\cdot x^k = 0.$$

$$\Rightarrow \sum_{k=2}^{\infty} k(k-1)c_k x^{k-1} + \sum_{k=2}^{\infty} 3k(k-1)c_k x^{k-2} - \sum_{k=1}^{\infty} 11k c_k x^{k-1} + \sum_{k=1}^{\infty} k c_k x^k + \sum_{k=0}^{\infty} c_k x^k = 0.$$

$$\Rightarrow \sum_{k=1}^{\infty} k(k+1)c_{k+1} x^k + \sum_{k=0}^{\infty} 3(k+1)(k+2) c_{k+2}\cdot x^k - \sum_{k=0}^{\infty} 11(k+1)c_{k+1} x^k + \sum_{k=1}^{\infty} k\cdot c_k x^k + \sum_{k=0}^{\infty} c_k\cdot x^k = 0.$$

$$\Rightarrow (c_0 - 11c_1 + 6c_2)x^0 + \sum_{k=1}^{\infty}\Big[c_k + kc_k - 11(k+1)c_{k+1} + k(k+1)c_{k+1} + 3(k+1)(k+2)c_{k+2}\Big]x^k = 0.$$

$$\Rightarrow c_0 - 11c_1 + 6c_2 + \sum_{k=1}^{\infty}\Big[(k+1)c_k + \big(k(k+1) - 11(k+1)\big)c_{k+1} + 3(k+1)(k+2)c_{k+2}\Big]x^k = 0.$$

$$\Rightarrow c_0 - 11c_1 + 6c_2 + \sum_{k=1}^{\infty}\Big[(k+1)c_k + (k+1)(k-11)c_{k+1} + 3(k+1)(k+2)c_{k+2}\Big]x^k = 0.$$

$$\therefore \boxed{1}\, c_{\boxed{0}} - \boxed{11}\, c_{\boxed{1}} + \boxed{6}\, c_{\boxed{2}} + \sum_{n=1}^{\infty}\Big[\boxed{(n+1)}\, c_{\boxed{n}} + \boxed{(n+1)(n-11)}\, c_{\boxed{n+1}} + \boxed{3(n+1)(n+2)}\, c_{\boxed{n+2}}\Big]x^n = 0.$$

(3)(a). From the constant term in the series above, we know that

$$c_0 - 11c_1 + 6c_2 = 0.$$

$$\Rightarrow 6c_2 = 11c_1 - c_0.$$

$$\Rightarrow c_2 = \frac{11}{6}c_1 - \frac{1}{6}c_0$$

$$\therefore c_{\boxed{2}} = \left(\boxed{11}\, c_{\boxed{1}} - c_{\boxed{0}}\right)/\boxed{6}$$

(b) From the series above, we find the recurrence relation is

$$(n+1)c_n + (n+1)(n-11)c_{n+1} + 3(n+1)(n+2)c_{n+2} = 0 , \; n \geq$$

$$\Rightarrow c_n + (n-11)c_{n+1} + 3(n+2)c_{n+2} = 0.$$

$$\Rightarrow 3(n+2)c_{n+2} = -(n-11)c_{n+1} - c_n.$$

$$\Rightarrow 3(n+2)c_{n+2} = (11-n)c_{n+1} - c_n.$$

$$\Rightarrow c_{n+2} = \frac{11-n}{3(n+2)}c_{n+1} - \frac{1}{3(n+2)}c_n.$$

$$\therefore c_{\boxed{n+2}} = \left(\boxed{(11-n)}\, c_{\boxed{n+1}} - c_{\boxed{n}}\right) \Big/ \boxed{3(n+2)} \quad \text{for} \quad \boxed{n} \geq \boxed{1}$$

(4) The general solution to $(x+3)y'' - (11-x)y' + y = 0$ converges atleast on $\boxed{(-3,\, 3)}$. and is

$$y(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots$$

From recurrence relation, we have

For $n=1$, $c_3 = \frac{10}{9}c_2 - \frac{1}{9}c_1 = \frac{10}{9}\left(\frac{11}{6}c_1 - \frac{1}{6}c_0\right) - \frac{1}{9}c_1.$

$$\Rightarrow c_3 = \frac{110}{54}c_1 - \frac{10}{54}c_0 - \frac{1}{9}c_1$$

$$\Rightarrow c_3 = \frac{55}{27}c_1 - \frac{5}{27}c_0 - \frac{1}{9}c_1$$

$$\Rightarrow c_3 = \frac{52}{27}c_1 - \frac{5}{27}c_0.$$

For $n=2$, $c_4 = \frac{9}{12}c_3 - \frac{1}{12}c_2.$

$$\Rightarrow c_4 = \frac{3}{4}\left(\frac{52}{27}c_1 - \frac{5}{27}c_0\right) - \frac{1}{12}\left(\frac{11}{6}c_1 - \frac{1}{6}c_0\right).$$

$$\Rightarrow c_4 = \frac{13}{9}c_1 - \frac{5}{36}c_0 - \frac{11}{72}c_1 + \frac{1}{72}c_0.$$

$$\Rightarrow c_4 = \frac{93}{72}c_1 - \frac{9}{72}c_0.$$

$\therefore c_4 = \frac{31}{24}c_1 - \frac{1}{8}c_0.$

So we have -

$$y(x) = c_0 + c_1 x + \left(\frac{11}{6}c_1 - \frac{1}{6}c_0\right)x^2 + \left(\frac{52}{27}c_1 - \frac{5}{27}c_0\right)x^3 + \left(\frac{31}{24}c_1 - \frac{1}{8}c_0\right)x^4 + \cdots$$

$$\Rightarrow y(x) = c_0\left[1 - \frac{1}{6}x^2 - \frac{5}{27}x^3 - \frac{1}{8}x^4 + \cdots\right] + c_1\left[x + \frac{11}{6}x^2 + \frac{52}{27}x^3 + \frac{31}{24}x^4 + \cdots\right]$$

$$\therefore y = c_0\left(\boxed{1} + \boxed{\frac{-1}{6}}x^2 + \boxed{\frac{-5}{27}}x^3 + \boxed{\frac{-1}{8}}x^4 + \cdots\right) + c_1\left(\boxed{1}x + \boxed{\frac{11}{6}}x^2 + \boxed{\frac{52}{27}}x^3 + \boxed{\frac{31}{24}}x^4 + \cdots\right)$$