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Differential Equations

Project 3

Population Dynamics and Food Supply: Exploring the Verhulst Model and the Predator-Prey Model

Introduction

Understanding population dynamics is critical for managing natural resources and ensuring ecosystem sustainability. Food supply is a significant component that drives population growth since it affects the population's ability to thrive and reproduce. We will look at two mathematical models that incorporate the function of food supply in population dynamics in this project: the Verhulst (logistic) model and the predator-prey model. We will also look at real-world examples to show how these models can be used to research and manage natural populations.

The Verhulst Model

The Verhulst model, commonly known as the logistic growth model, is a differential equation that models population expansion when resources are restricted. The model considers the environment's carrying capacity (K), which is the maximum population size that the environment can support. The Verhulst model is expressed as follows:

$$dP/dt = rP(1 - P/K)$$

Where P represents the population size, t is the time, r is the intrinsic growth rate, and K is the carrying capacity. The solution of this equation provides a function P(t) that describes the population changes over time.

Fish population in a pond

Consider a pond with a fish population, where the fish have limited food resources. Suppose the intrinsic growth rate (r) is 0.8 per year, and the carrying capacity (K) is 2000 fish.

The Verhulst model for this scenario is:

$$dP/dt = 0.8P(1 - P/2000)$$

By solving this differential equation, we obtain the population function P(t):

$$P(t) = (2000 * P0 * e^{(0.8t)}) / (2000 + P0 * (e^{(0.8t)} - 1))$$

Where P0 is the initial population size and e is the base of the natural logarithm.

Using this model, we can predict the fish population's growth over time and determine how the population size will stabilize as it approaches the carrying capacity.

The Predator-Prey Model

The predator-prey model, often represented by the Lotka-Volterra equations, is a system of two first-order differential equations that describe the interaction between predator and prey populations. The model incorporates the effects of food supply on both populations. The Lotka-Volterra equations can be written as:

$$dx/dt = \alpha x - \beta xy$$

$$dy/dt = \delta xy - \gamma y$$

In these equations, α, β, γ, and δ are positive constants representing the prey's intrinsic growth rate, the rate at which predators consume prey, the predator's death rate, and the efficiency of converting consumed prey into predator reproduction, respectively. The solutions to this system of equations provide functions $x(t)$ and $y(t)$ that describe the population dynamics of the predator and prey over time.

Snowshoe hare and Canadian lynx populations

A well-known example of predator-prey dynamics is the interaction between snowshoe hare and Canadian lynx populations. Suppose the hare's intrinsic growth rate (α) is 0.6 per year, the rate at which lynx consume hares (β) is 0.025 per year, the lynx's death rate (γ) is 0.4 per year, and the efficiency of converting consumed hares into lynx reproduction (δ) is 0.015.

The Lotka-Volterra equations for this scenario are:

$$dx/dt = 0.6x - 0.025xy$$

$$dy/dt = 0.015xy - 0.4y$$

We can use numerical methods to solve this system of differential equations and obtain the population functions $x(t)$ and $y(t)$ that describe the dynamics of the snowshoe hare and Canadian lynx populations over time. Due to the complexity of the system, it is challenging to find a closed-form solution.

By plotting the solutions x(t) and y(t) against time (t), we can observe the oscillatory behavior of the hare and lynx populations. As the hare population grows, it provides more food for the lynx population, causing it to increase as well. However, as the lynx population grows, predation pressure on the hares increases, leading to a decline in the hare population. As the hare population decreases, the lynx population declines due to a lack of food, and the cycle repeats.

Conclusion

In this project, we explored two mathematical models that incorporate food supply in population dynamics: the Verhulst model and the predator-prey model. These models help us understand the complex relationships between population growth and food supply, and can inform decision-making in fields such as ecology, conservation, and natural resource management. By studying real-world examples, such as fish population growth in a pond and the interaction between snowshoe hare and Canadian lynx populations, we demonstrated the applicability of these models in understanding and addressing real-world challenges. With a deeper understanding of these mathematical models, we can work towards managing ecosystems and natural resources more effectively, promoting a sustainable future for all living beings.

References

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