Test #2 (Version A)

4a) Consider the nonhomogenous differential equation. Find a fundamental set $\{y_1, y_2\}$ of solutions of the associated homogenous equation.

$$y'' + 15y' + 50y = \sin(e^{5x})$$

Homogenous equation: $y'' + 15y' + 50y = 0$

$ar^2 + br + c = 0$ $\qquad y_1 = e^{r_1 x} \quad y_2 = e^{r_2 x}$

$r^2 + 15r + 50 = 0$ $\qquad \boxed{y_1 = e^{-10x} \quad y_2 = e^{-5x}}$

$(r+10)(r+5) = 0$

$r + 10 = 0 \quad | \quad r + 5 = 0$

$r_1 = -10 \quad | \quad r_2 = -5$

4b) Use variation of parameters to find a particular solution y_p of the above nonhomogenous equation.

$y_p = u_1 y_1 + u_2 y_2$

$y_p = u_1 e^{-10x} + u_2 e^{-5x}$

$y_p' = u_1' e^{-10x} - 10u_1 e^{-10x} + u_2' e^{-5x} - 5u_2 e^{-5x}$

$y_p' = \underbrace{u_1' e^{-10x} + u_2' e^{-5x}}_{=0} - 10u_1 e^{-10x} - 5u_2 e^{-5x}$

$y_p' = -10u_1 e^{-10x} - 5u_2 e^{-5x}$

$y_p'' = -10(u_1' e^{-10x} - 10u_1 e^{-10x}) - 5(u_2' e^{-5x} - 5u_2 e^{-5x})$

$y_p'' = -10u_1' e^{-10x} + 100u_1 e^{-10x} - 5u_2' e^{-5x} + 25u_2 e^{-5x}$

$LS = y_p'' + 15y_p' + 50y_p$

$= -10u_1' e^{-10x} + 100u_1 e^{-10x} - 5u_2' e^{-5x} + 25u_2 e^{-5x} + 15(-10u_1 e^{-10x} - 5u_2 e^{-5x}) + 50y_p$

$= -10u_1' e^{-10x} + \cancel{100u_1} e^{-10x} - 5u_2' e^{-5x} + \cancel{25u_2 e^{-5x}} - \cancel{150u_1} e^{-10x} - \cancel{75u_2} e^{-5x} + \cancel{50u_1} e^{-10x} + \cancel{50u_2}$

$= -10u_1' e^{-10x} - 5u_2' e^{-5x}$

$RS = \sin(e^{5x})$

Need $LS = RS$

$-10u_1' e^{-10x} - 5u_2' e^{-5x} = \sin(e^{5x})$

$\underline{10(u_1' e^{-10x} + u_2' e^{-5x} = 0)}$

$$-10\cancel{u'_1}e^{-10x} - 5u'_2e^{-5x} = \sin(e^{5x})$$
$$+\ \cancel{10u'_1}e^{-10x} + 10u'_2e^{-5x} = 0$$

$$\frac{5u'_2\cancel{e^{-5x}}}{\cancel{e^{-5x}}} = \frac{\sin(e^{5x})}{e^{-5x}}$$

$$\frac{\cancel{5}u'_2}{\cancel{5}} = \frac{e^{5x}\sin(e^{5x})}{5}$$

$$u'_2 = \frac{e^{5x}\sin(e^{5x})}{5}$$

$$u_2 = \int \tfrac{1}{5}e^{5x}\sin(e^{5x})\,dx$$

$$u_2 = -\frac{\cos(e^{5x})}{25} + c_2$$

$$u'_1e^{-10x} + u'_2e^{-5x} = 0$$

$$u'_1e^{-10x} + \left(\frac{e^{5x}\sin(e^{5x})}{5}\right)e^{-5x} = 0$$

$$u'_1e^{-10x} + \frac{\sin(e^{5x})}{5} = 0$$

$$u'_1e^{-10x} = -\frac{\sin(e^{5x})}{5}$$

$$u'_1 = -\frac{\sin(e^{5x})}{5e^{-10x}}$$

$$u_1 = \int -\frac{\sin(e^{5x})}{5e^{-10x}}\,dx$$

$$u_1 = \frac{e^{5x}\cos(e^{5x}) - \sin(e^{5x})}{25} + c_1$$

$$y_p = u_1e^{-10x} + u_2e^{-5x}$$

$$y_p = \left(\frac{e^{5x}\cos(e^{5x}) - \sin(e^{5x})}{25} + c_1\right)e^{-10x} + \left(-\frac{\cos(e^{5x})}{25} + c_2\right)e^{-5x}$$

$$\boxed{y_p = -\frac{e^{-10x}\sin(e^{5x})}{25}}$$

4c) Use your previous answers to determine the general solution of the above nonhomogenous equation

$$y = y_p + c_1y_1 + c_2y_2$$

$$\boxed{y = -\frac{e^{-10x}\sin(e^{5x})}{25} + c_1e^{-10x} + c_2e^{-5x}}$$