Section 4.3: Elementary Mechanics

3) A boat weighs 64,000 lb. Its propeller produces a constant thrust of 50,000 lb and the water exerts a resistive force with magnitude proportional to the speed, with $k = 2000$ lb-s/ft. Assuming that the boat starts from rest, find its velocity as a function of time, and find its terminal velocity.

What we know:

* Weight of the boat = 64,000 lb
* Thrust (Force) produced by the propeller = 50,000 lb
* Resistive force exerted by water is proportial to the boat's speed (velocity).

Therefore:

constant

Velocity

$$F_R = k \cdot v$$

Resistive Force

$k = 2000$ lb-s/ft

$F_R = 2000v$

The total force or Net force acting on the boat = propeller thrust − resistive force

total force → $F = 50{,}000 - 2000v$

We know the weight of the boat is 64,000 lb, so we need to find the mass of the boat.

$$W = mg$$

* W → weight
* m → mass
* g = 32

$$m = \frac{W}{g}$$

$$m = \frac{64{,}000}{32}$$

$$m = 2000 \text{ lb}$$

We also know that force = mass × acceleration ($F = ma$)

$$a = v'$$

$$F = ma$$

$$\frac{50{,}000 - 2000v}{2000} = \frac{20\cancel{0}0v'}{20\cancel{0}0}$$

$$\frac{50{,}000}{2000} - \frac{20\cancel{0}0v}{2\cancel{0}00} = v'$$

$$25 - v = v'$$

$$v' = -(v-25)$$

$$\frac{dv}{dt} = -(v-25)$$

$$\frac{dv}{v-25} = -dt$$

$$\int \left(\frac{1}{v-25}\right) dv = \int -dt$$

$$\log|v-25| = -t + c$$

$$e^{\log|v-25|} = e^{-t+c}$$

$$e^{\log|v-25|} = e^{-t} \cdot e^{c}$$

$$v - \cancel{25} = ce^{-t}$$
$$+\cancel{25} \qquad +25$$

$$v = ce^{-t} + 25$$

The boat started from rest. So $v(0)=0$. At time $t=0$, the boat does not move at all.

$$v(0) = ce^{-(0)} + 25$$

$$v(0) = c + 25$$

$$0 = c + \cancel{25}$$
$$-25 \qquad -\cancel{25}$$

$$c = -25$$

We can rewrite the equation:

$$V = ce^{-t} + 25$$

$$V = -25e^{-t} + 25$$

$$\boxed{V = 25(-e^{-t} + 1)\ ft/s}$$ ← Velocity as a function of time

Now we need to find the terminal velocity. (Maximum Velocity)

$$V = 25(-e^{-t} + 1)$$

$$V' = 25\left(\frac{d}{dt}(-e^{-t}) + \frac{d}{dt}(1)\right)$$

$$V' = 25(e^{-t} + 0) = 25e^{-t}$$

$$V'' = 25\left(\frac{d}{dt}(e^{-t})\right) = 25(-e^{-t}) = -25e^{-t}$$

We can conclude that v is increasing for $v' = 25e^{-t} > 0$ and $v'' < 0$. Therefore the boat would approach its maximum velocity as $t \to \infty$ (as time keeps increasing).

$$\text{Terminal velocity} = \lim_{t\to\infty} 25(-e^{-t} + 1)$$

$$= 25\left(-\lim_{t\to\infty} e^{-t} + 1\right)$$

$$= 25(1 - 0) = \boxed{25\ ft/s}$$ ← terminal velocity